第 I 部分 选择题 (15×3 分)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	D	E	C	B	B	B	E	E	D	C	D	C	A	C

Question 1:

You may find the following useful in this question:

在本题中可能会用到：

(1) For small angle θ,

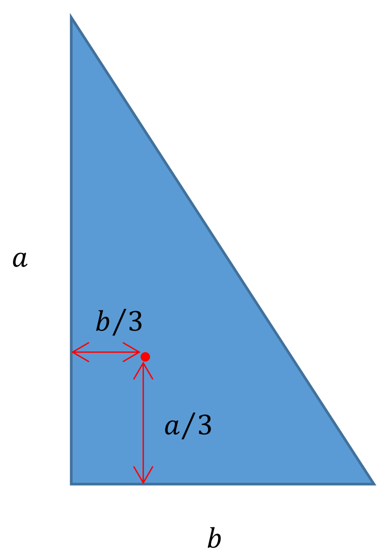
当θ为小角度时，

$$\cos\theta \approx \sec\theta \approx 1$$

$$\sin\theta \approx \tan\theta \approx \theta$$

(2) The centroid of a right triangle is located as shown below:‘

直角三角形的形心位置如下所示：



Consider a hollow cubic box with side length L. The walls of the box have negligible thickness and uniform mass density, with mass per unit area σ. Now the top wall of the box is removed and the box is put with its bottom wall facing downwards into a fluid (with mass per unit volume ρ).

Note: We assume the fluid is ideal and in static equilibrium, such that dynamic effects like fluid inertia and viscosity are negligible. The only fluid force is the hydrostatic buoyant force.

考虑一个边长为 L 的空心立方体箱子。箱子的边面厚度可以忽略不计，且具有均匀的质量密度，每单位面积的质量为σ。现在，箱子的顶面被去掉，箱子底面朝下放入一流体中（流体的每单位体积质量为ρ）。

注意：我们假设流体是理想的，处于静态平衡中，因此流体惯性和粘度等动态效应可以忽略。唯一的流体力为静浮力。

(a) Find the critical surface mass density of the walls σ_c above which the box will immerse completely into the fluid. Express your answer in terms of ρ and L.

找到边面的临界表面质量密度σ_c，超过这个值箱子将完全浸入流体中。用 ρ 和 L 表达你的答案。

Hereafter, we assume $\sigma < \sigma_c$.

以下均假设 $\sigma < \sigma_c$。

(b) What is the height of the box, h, that will be beneath the fluid surface? Express your answer in terms of σ and ρ.

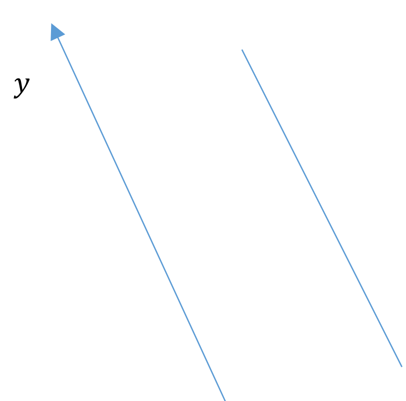
箱子在流体表面下方的高度 h 是多少？用σ和ρ表达你的答案。

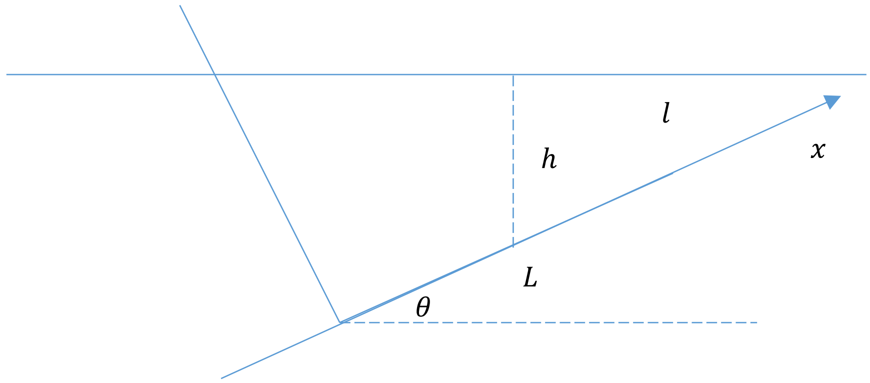
(c) What is the frequency of small vertical oscillations of the box at its equilibrium position? Express your answer in terms of ρ, σ and g.

箱子在其平衡位置的小幅垂直振荡的频率是多少？用ρ、σ和引力加速度 g 表达你的答案。

Now the box is tilted by an angle of θ about a horizontal axis, which passes through its center of mass (CM) and is perpendicular to two vertical faces.

现在箱子绕一水平轴倾斜了一个角度θ，该轴通过其质心，并且垂直于两个悬直边面。





(d) Find the moment of inertia of the box about the axis described above. Express your answer in terms of σ and L.

找到箱子对于上述转轴之转动惯量。用σ和 L 表达你的答案。

When the center of the bottom surface is at a distance h below the fluid surface, the net external force acting on the box is zero.

当底面中心距离流体表面的深度为 h 时，作用于箱子的净外力为零。

(e) Find h and the depth of the CM of the box beneath the surface. Express your answers in terms of σ, ρ, and θ. What are the two respective expressions for small angle θ.

找到 h 以及箱子质心在表面下方的深度。用σ、ρ和θ表达你的答案。并分别写出当 θ为小角度时的表达式。

(f) Find the location of the geometric center of the volume of the box below the fluid surface. Express you answer in terms of L, h, and θ.

找到流体表面下方盒子体积的几何中心位置。用 L、h 和θ表达你的答案。

Assume that the initial angle the box is tilted is small and the initial h has the value in part (e) for small angle. The box is then released to rotate freely.

假设盒子初始倾斜角度很小，初始高度 h 在小角度情况下的值为(e)部分的值。然后盒子被释放以自由旋转。

(g) Find the horizontal and vertical net force acting on the box under small angle rotation by keeping terms up to first order in θ.

在小角度旋转时，通过仅保留到θ的一阶项，找出作用在盒子上的水平和垂直合力。

(h) Find the torque acting on the box for small θ by keeping only terms up to first order in θ. Express your answer in terms of σ, ρ, g, L, and θ.

通过仅保留到θ的一阶项，找到作用在盒子上的扭矩。用σ、ρ、g、L和θ表达你的答案。

(i) Show that the box will undergo simple harmonic angular oscillations and find the frequency.

证明盒子将作简谐角振动，并找出其频率。

(j) For fixed L, find the condition under which the frequency is minimum. What is this minimum frequency in terms of g and L?

在给定 L 的情况下，找出频率为最小的条件，并以 g 和 L 表达该最小频率。

Solution

(a)

$$\sigma_c 5L^2 = \rho L^3$$

$$\sigma_c = \frac{\rho L}{5}$$

(b)

$$\sigma 5L^2 = \rho L^2 h$$

$$h = \frac{5\sigma}{\rho}$$

(c)

$$F = -\rho g L^2 \Delta h$$

$$f = \frac{1}{2\pi}\sqrt{\frac{\rho g L^2}{\sigma 5 L^2}} = \frac{1}{2\pi}\sqrt{\frac{\rho g}{5\sigma}}$$

(d) The moment of inertia of the box about the rotational axis passing through CM is

$$\left(\frac{1}{12}\sigma L^2 \times L^2 + \sigma L^2 \times \left(\frac{2L}{5}\right)^2\right) + 2\left(\frac{1}{12}\sigma L^2 \times L^2 + \sigma L^2\left(\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2} - \frac{2L}{5}\right)^2\right)\right)$$
$$+ 2\left(\frac{1}{12}\sigma L^2 \times 2L^2 + \sigma L^2\left(\frac{L}{2} - \frac{2L}{5}\right)^2\right) = \frac{77}{60}\sigma L^4$$

(e)

$$\rho g L \times Lh \sec\theta = \rho g L^2 h \sec\theta$$

$$\sigma g 5 L^2 = \rho g L^2 h \sec\theta$$

$$h = \frac{5\sigma}{\rho}\cos\theta$$

The depth of the CM of the box is

$$\left(h \sec\theta - \frac{2L}{5}\right)\cos\theta = \left(\frac{5\sigma}{\rho} - \frac{2L}{5}\right)\cos\theta$$

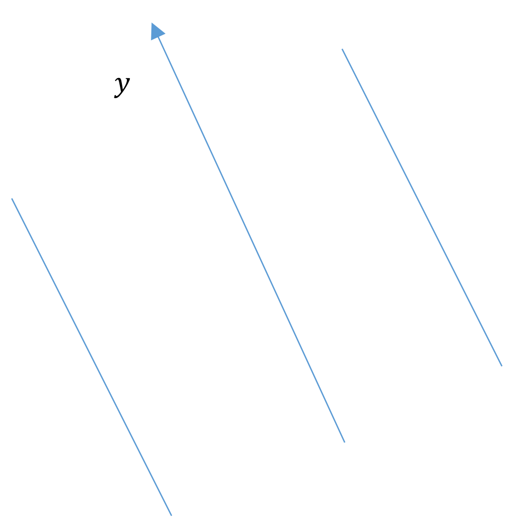
For small θ,

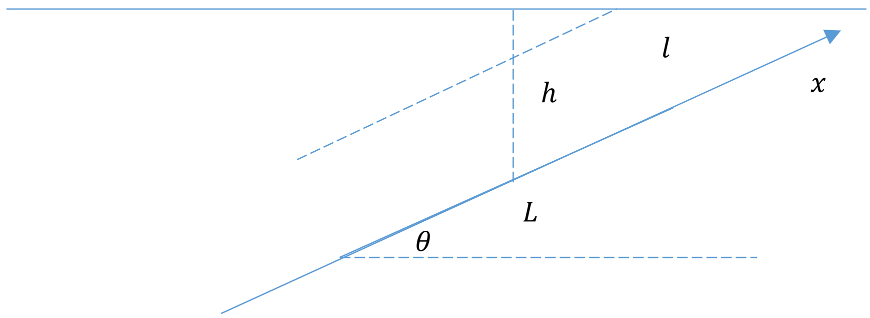
$$h = \frac{5\sigma}{\rho}$$

and the depth of the CM is

$$\frac{5\sigma}{\rho} - \frac{2L}{5}$$

(f)





Solve for l:

$$h = \frac{L}{2}\sin\theta + l\cos\theta$$

$$l = h\sec\theta - \frac{L}{2}\tan\theta$$

The center of mass of the right angle triangle is at

$$\left(-\frac{L}{2}+\frac{L}{3}, l+\frac{L}{3}\tan\theta\right) = \left(-\frac{L}{6}, h\sec\theta - \frac{L}{6}\tan\theta\right)$$

The area of the right angle triangle is

$$\frac{L^2\tan\theta}{2}$$

The center of mass of the rectangle is at

$$\left(0,\frac{l}{2}\right) = \left(0,\frac{h}{2}\sec\theta - \frac{L}{4}\tan\theta\right)$$

The area of the rectangle is

$$Ll = Lh\sec\theta - \frac{L^2}{2}\tan\theta$$

The total area is

$$Lh\sec\theta$$

The center of mass of the part below the fluid surface is

$$\left[\frac{L^2\tan\theta}{2}\left(-\frac{L}{6}, h\sec\theta - \frac{L}{6}\tan\theta\right) + \left(Lh\sec\theta - \frac{L^2}{2}\tan\theta\right)\left(0,\frac{h}{2}\sec\theta - \frac{L}{4}\tan\theta\right)\right]\Big/ Lh\sec\theta$$

$$= \frac{\left(-\frac{L}{6}\frac{L^2\tan\theta}{2}, \frac{L^2\tan\theta}{2}\left(h\sec\theta - \frac{L}{6}\tan\theta\right) + \left(Lh\sec\theta - \frac{L^2}{2}\tan\theta\right)\left(\frac{h}{2}\sec\theta - \frac{L}{4}\tan\theta\right)\right)}{Lh\sec\theta}$$

$$= \frac{\left(-\frac{L^3\tan\theta}{12}, \frac{L^3\tan^2\theta}{24} + \frac{Lh^2\sec^2\theta}{2}\right)}{Lh\sec\theta}$$

$$= \left(-\frac{L^2}{12h}\sin\theta, \frac{L^2}{24h}\tan^2\theta\cos\theta + \frac{h}{2}\sec\theta\right)$$

(g) In horizontal direction there is no net force. In the vertical direction,

$$F = \sigma g 5L^2 - \rho g L^2 h\sec\theta \approx \sigma g 5L^2 - \rho g L^2 h = 0$$

(h) The center of mass of the box is at $\left(0, \frac{2L}{5}\right)$.

The horizontal position of the CM in (f) w.r.t. the CM of the box (right = positive) is

$$-\frac{L^2}{12h}\sin\theta\cos\theta - \left(\frac{L^2}{24h}\tan^2\theta\cos\theta + \frac{h}{2}\sec\theta - \frac{2L}{5}\right)\sin\theta$$

$$\approx -\frac{L^2}{12h}\theta - \left(\frac{h}{2} - \frac{2L}{5}\right)\theta$$

$$= -\left(\frac{L^2}{12h} + \frac{h}{2} - \frac{2L}{5}\right)\theta$$

$$= -\left(\frac{L^2}{60\,\sigma/\rho} + \frac{5}{2}\frac{\sigma}{\rho} - \frac{2L}{5}\right)\theta$$

$$= -\frac{L}{60}\left(\frac{\rho L}{\sigma} + 150\frac{\sigma}{\rho L} - 24\right)\theta$$

The torque (counterclockwise = positive) is

$$\tau = -\sigma g 5L^2\frac{L}{60}\left(\frac{\rho L}{\sigma} + 150\frac{\sigma}{\rho L} - 24\right)\theta$$

$$= -\frac{1}{12}\sigma g L^3\left(\frac{\rho L}{\sigma} + 150\frac{\sigma}{\rho L} - 24\right)\theta$$

(i) Since

$$\frac{\rho L}{\sigma} + \frac{150}{\rho L/\sigma} - 24 = \sqrt{150}\left(\frac{\rho L/\sigma}{\sqrt{150}} + \frac{\sqrt{150}}{\rho L/\sigma}\right) - 24 \geq 2\sqrt{150} - 24 = 10\sqrt{6} - 24 > 0$$

hence it will undergo simple harmonic angular oscillations.

The frequency is

$$\frac{1}{2\pi}\sqrt{\frac{\frac{1}{12}\sigma g L^3\left(\frac{\rho L}{\sigma}+150\frac{\sigma}{\rho L}-24\right)}{\frac{77}{60}\sigma L^4}}=\frac{1}{2\pi}\sqrt{\frac{5}{77}\left(\frac{\rho L}{\sigma}+150\frac{\sigma}{\rho L}-24\right)\frac{g}{L}}$$

(j) Frequency is minimum when

$$\frac{\rho L/\sigma}{\sqrt{150}}=\frac{\sqrt{150}}{\rho L/\sigma}\Rightarrow\frac{\rho L}{\sigma}=5\sqrt{6}$$

The minimum frequency is

$$\frac{1}{2\pi}\sqrt{\frac{5}{77}\left(2\sqrt{150}-24\right)\frac{g}{L}}=\frac{1}{2\pi}\sqrt{\frac{10}{77}\left(5\sqrt{6}-12\right)\frac{g}{L}}$$

MC

1. A cannon on the ground can fire cannon balls with a speed of 50.0 m/s at any angle. A drone is flying at a constant height of 100 m above ground. What is the horizontal distance from the cannon beyond which the cannon cannot take down the drone?

地面上的一门火炮可以以 50.0 m/s 的速度发射炮弹，角度任意。一架无人机正在离地面 100 米 的恒定高度飞行。问在火炮水平距离多远以外的地方，火炮无法击落无人机？

A. 119 m

B. 123 m

C. 136 m

D. 141 m

E. 157 m

Solution

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta = x\tan\theta - \frac{gx^2}{2v^2}(1+\tan^2\theta)$$

$$\tan^2\theta - \frac{2v^2}{gx}\tan\theta + 1 + \frac{2v^2 y}{gx^2} = 0$$

No solution of $\tan\theta$ if

$$\left(-\frac{2v^2}{gx}\right)^2 - 4\left(1 + \frac{2v^2 y}{gx^2}\right) < 0$$

$$\frac{v^4}{g^2} - \frac{2v^2 y}{g} < x^2$$

$$|x| > \sqrt{\frac{v^4}{g^2} - \frac{2v^2 y}{g}} = 119 \text{ m}$$

2. A sound source and a listener are moving towards each other. The speeds of the source and the listener are 30.0 m/s and 20.0 m/s, respectively. There is also a strong wind blowing at a speed of 10.0 m/s in the direction from the listener to the

source. If the source emits sound waves with a frequency of 440 Hz, what is the frequency measured by the listener? Take the speed of sound in air to be 340 m/s.

一个声源和一个听者正向着彼此移动。声源的速率为 30.0 m/s，听者的速率为 20.0 m/s。同时，有一股速率 10.0 m/s 的强风从听者吹向声源方向。如果声源发出频率为 440 Hz 的声音波，那么听者测得的频率是多少？取空气中声音的速度为 340 m/s。

A. 507 Hz

B. 509 Hz

C. 511 Hz

D. 513 Hz

E. 515 Hz

Solution

In the air frame, the listener is moving with speed 10.0 m/s towards the source and the source is moving with speed 40.0 m/s towards the listener. Hence

$$f_L = \frac{340 + 10.0}{340 - 40.0} f_S = \frac{7}{6} \times 440 = 513 \text{ Hz}$$

3. The latitude of Hong Kong is 22.3°N. Assume that the Earth is perfectly spherical with a radius of 6400 km, and has a self-rotational period of 24 hours exactly. A 1.00-kg object on the ground is at rest (i.e., rotating together with the Earth). What is the magnitude of friction exerted by the ground on it?

香港的纬度为北纬 22.3°。假设地球是一个完美球体，半径为 6400 公里，并且自转周期正好是 24 小时。一个质量为 1.00 公斤的物体静止在地面上（即与地球一起旋转）。地面施加于该物体上的摩擦力有多大？

A. 0.0036 N

B. 0.0058 N

C. 0.0074 N

D. 0.0093 N

E. 0.012 N

Solution

$$N\sin 22.3^{\circ} + f\cos 22.3^{\circ} = mg\sin 22.3^{\circ}$$

$$(mg - N)\sin 22.3^{\circ} = f\cos 22.3^{\circ}$$

$$f\sin 22.3^{\circ} + mg\cos 22.3^{\circ} - N\cos 22.3^{\circ} = m\omega^2 R\cos 22.3^{\circ}$$

$$f\sin 22.3^{\circ} + f\cot 22.3^{\circ}\cos 22.3^{\circ} = m\omega^2 R\cos 22.3^{\circ}$$

$$f = m\omega^2 R\cos 22.3^{\circ}\sin 22.3^{\circ}$$

$$\frac{f}{m} = \omega^2 R\cos 22.3^{\circ}\sin 22.3^{\circ} = 0.012\ \text{N}$$

4. A 0.10-kg particle is attached to a spring with spring constant $k = 10$ N/m to undergo simple harmonic motion on the x-axis with equilibrium position at $x = 0$. At $t = 0$, the particle is at $x_0 = -0.40$ m moving with velocity $v_0 = 3.0$ m/s. Find its position at $t = 2.0$ s.

一个质量为 0.10 公斤 的粒子连接到弹簧常数 $k = 10$ N/m 的弹簧上，在 x 轴上做简谐运动，平衡位置在 $x = 0$。当 $t = 0$ 时，该粒子位于 $x_0 = -0.40$ m 处，并以速度 $v_0 = 3.0$ m/s 移动。求其在 $t = 2.0$ s 时的位置。

A. –0.27 m

B. –0.11 m

C. 0.11 m

D. 0.25 m

E. 0.27 m

Solution

$$\omega = \sqrt{\frac{10}{0.10}} = 10$$

$$A\cos\phi = -0.40$$

$$-\omega A\sin\phi = 3.0$$

$$A\sin\phi = -0.30$$

$$\tan\phi = 0.75$$

$$\phi = \tan^{-1} 0.75 + \pi = 3.785$$

$$A = \sqrt{0.40^2 + 0.30^2} = 0.50$$

$$x(2.0) = 0.50\cos(10 \times 2.0 + 3.785) = 0.11 \text{ m}$$

5. A 1.00-kg mass and a 2.00-kg mass are attached to the two ends of a spring with spring constant k = 20.0 N/m. The spring is initially at its natural length and the two masses are given the same initial speed of 3.00 m/s moving away from each other. Find the maximum extension of the spring.

   一个 1.00 公斤的物体和一个 2.00 公斤的物体分别连接在弹簧的两端，弹簧的弹性系数 k 为 20.0 N/m。弹簧初始时处于自然长度，两物体以大小同为 3.00 m/s 的初速率朝彼此远离的方向运动。求弹簧的最大伸长量。

   A. 0.875 m

   B. 1.10 m

   C. 1.12 m

   D. 1.14 m

   E. 1.16 m

Solution

$$3.00v = 2.00 \times 3.00 - 1.00 \times 3.00 = 3.00 \Rightarrow v = 1.00$$

$$\frac{1}{2} \times 3.00 \times 1.00^2 + \frac{1}{2}ke^2 = \frac{1}{2} \times 3.00 \times 3.00^2$$

$$10.0e^2 = 12.0 \Rightarrow e = \sqrt{1.20} = 1.10 \text{ m}$$

6. A transverse wave on a string is moving to the right. The graph below of $y(x, t)$ versus coordinate x for a specific time t shows the shape of part of the string at that time. Point 1 is a local maximum and point 4 is a local minimum. At this time, the instantaneous velocity of a particle on the string is upward at

   一条绳子上的横波向右移动。下图显示了在特定时间 t 下 $y(x, t)$ 与坐标 x 的关系，表示那时绳子一部分的波形。点 1 是局部最大值，点 4 是局部最小值。在这个时刻，绳子上哪些标示的粒子的瞬时速度为向上？

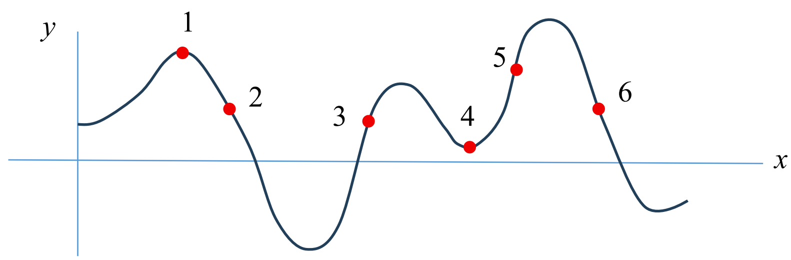
   A. 只 1 和 4

B. 只 2 和 6

C. 只 3 和 5

D. 只 1, 3 和 5

E. 只 2, 4 和 6



7. An object moving on the *xy* plane is under a force given by

$$\vec{F}(x,y) = Cy\hat{\imath}$$

where *C* is a constant and $\hat{\imath}$ is the unit vector in the positive *x* directions.

Consider the square path with side length *L* and the four corners at (0, 0), (*L*, 0), (0, *L*), and (*L*, *L*). The object starts from the origin and moves along a square path to reach the corners in the order (0, 0) → (*L*, 0) → (*L*, *L*) → (0, *L*) → (0, 0). Find the work done by the force.

在 *xy* 平面上移动的物体受到的力为

$$\vec{F}(x,y) = Cy\hat{\imath}$$

其中 *C* 是一个常数，$\hat{\imath}$ 正 *x* 方向的单位向量。

考虑一个边长为 *L* 的正方形路径，四个角分别位于 (0, 0)、(*L*, 0)、(0, *L*) 和 (*L*, *L*)。物体从原点开始，沿着正方形路径移动，依次到达角点 (0, 0) → (*L*, 0) → (*L*, *L*) → (0, *L*) → (0, 0)。求该力所做的功。

A. CL^2

B. $-CL^2$

C. 0

D. $2CL^2$

E. $-2CL^2$

Questions 8 and 9

A spring-loaded object of mass m is launched at $t = 0$ with speed u at an angle that makes an angle of 45° with the horizontal. The object would hit the ground again at $t = T$. However, at $t = T/4$, the spring releases and the object breaks into two equal-mass pieces, in such a way that one piece will travel backwards along the same trajectory to hit the launcher.

问题 8 和 9

一个质量为 m 内置弹簧的发射物在 $t = 0$ 时以速率 u 被发射，发射角与水平方向成 45°。该物体本该在 $t = T$ 时再次落地。然而，在 $t = T/4$ 时，弹簧释放，物体分裂成两个相等质量的部分，其中一部分将沿着相同的轨迹向后移动，撞击发射器。

8. What is the energy released by the spring?

弹簧释放的能量是多少？

A. $\frac{1}{4}mu^2$

B. $\frac{1}{2}mu^2$

C. $\frac{3}{4}mu^2$

D. mu^2

E. $\frac{5}{4}mu^2$

Solution

$$\frac{T}{4} = \frac{u\sin 45^\circ}{2g} = \frac{u}{2\sqrt{2}g}$$

$$v_x\left(\frac{T}{4}\right) = u\cos 45^\circ = \frac{u}{\sqrt{2}}$$

$$v_y\left(\frac{T}{4}\right) = u\sin 45^\circ - g\frac{T}{4} = \frac{u}{\sqrt{2}} - g\frac{u}{2\sqrt{2}g} = \frac{u}{2\sqrt{2}}$$

$$v\left(\frac{T}{4}\right) = \sqrt{\frac{u^2}{2} + \frac{u^2}{8}} = \frac{\sqrt{5}}{2\sqrt{2}}u$$

$$U = \frac{1}{2}\frac{m}{2}v^2 + \frac{1}{2}\frac{m}{2}(3v)^2 - \frac{1}{2}mv^2 = 2mv^2 = 2m\frac{5}{8}u^2 = \frac{5}{4}mu^2$$

9. What is the distance between the two pieces when the launcher is hit?

当发射器被击中时，两部分之间的距离是多少？

A. $\frac{1}{2}\frac{u^2}{g}$

B. $\frac{1}{\sqrt{2}}\frac{u^2}{g}$

C. $\frac{\sqrt{3}}{2}\frac{u^2}{g}$

D. $\frac{u^2}{g}$

E. $\frac{\sqrt{5}}{2}\frac{u^2}{g}$

Solution:

(i) Direct method

$$x\left(\frac{T}{4}\right) = u\frac{T}{4}\cos 45^\circ = \frac{u^2}{4g}$$

$$y\left(\frac{T}{4}\right) = u\frac{T}{4}\sin 45^\circ - \frac{1}{2}g\left(\frac{T}{4}\right)^2 = \frac{u^2}{4g} - \frac{u^2}{16g} = \frac{3u^2}{16g}$$

$$x\left(\frac{T}{2}\right) = \frac{u^2}{4g} + 3\frac{u}{\sqrt{2}}\frac{T}{4} = \frac{u^2}{4g} + 3\frac{u}{\sqrt{2}}\frac{u}{2\sqrt{2}g} = \frac{u^2}{4g} + \frac{3u^2}{4g} = \frac{u^2}{g}$$

$$y\left(\frac{T}{2}\right) = \frac{3u^2}{16g} + 3\frac{u}{2\sqrt{2}}\frac{T}{4} - \frac{1}{2}g\left(\frac{T}{4}\right)^2 = \frac{3u^2}{16g} + 3\frac{u}{2\sqrt{2}}\frac{u}{2\sqrt{2}g} - \frac{1}{2}g\left(\frac{u}{2\sqrt{2}g}\right)^2$$

$$= \frac{3u^2}{16g} + \frac{3u^2}{8g} - \frac{u^2}{16g} = \frac{u^2}{2g}$$

$$d=\sqrt{\left(\frac{u^2}{g}\right)^2+\left(\frac{u^2}{2g}\right)^2}=\frac{\sqrt{5}}{2}\frac{u^2}{g}$$

(ii) CM method

$$\frac{T}{2}=\frac{u\sin 45^\circ}{g}=\frac{u}{\sqrt{2}g}$$

$$x_{CM}\left(\frac{T}{2}\right)=u\frac{T}{2}\cos 45^\circ=\frac{u^2}{2g}$$

$$y_{CM}\left(\frac{T}{2}\right)=u\frac{T}{2}\sin 45^\circ-\frac{1}{2}g\left(\frac{T}{2}\right)^2=\frac{u^2}{2g}-\frac{u^2}{4g}=\frac{u^2}{4g}$$

$$d=2\sqrt{\left(\frac{u^2}{2g}\right)^2+\left(\frac{u^2}{4g}\right)^2}=\sqrt{\left(\frac{u^2}{g}\right)^2+\left(\frac{u^2}{2g}\right)^2}=\frac{\sqrt{5}}{2}\frac{u^2}{g}$$

(iii) Relative motion method

The relative speed is $4v$ with no relative acceleration.

Hence

$$d=4v\frac{T}{4}=4\frac{\sqrt{5}}{2\sqrt{2}}u\frac{u}{2\sqrt{2}g}=\frac{\sqrt{5}}{2}\frac{u^2}{g}$$

10. A solid cylinder with mass M and radius R, rotating with angular speed ω_0 about an axis through its center, is set on a horizontal surface of which the kinetic friction coefficient is μ_k. The initial linear velocity of the cylinder zero. Calculate the distance the cylinder rolls as it moves from where it was set down to where it begins to roll without slipping.

一个质量为 M、半径为 R、 围绕其中心轴以角速度ω_0旋转的实心圆柱体被放置在一个动摩擦系数为μ_k的水平面上。圆柱体的初始线速度为零。计算圆柱体从放置的位置移动到开始无滑动滚动的位置之间的距离。

A. $\frac{\omega_0^2R^2}{12\mu_k g}$

B. $\frac{\omega_0^2R^2}{14\mu_k g}$

C. $\frac{\omega_0^2R^2}{16\mu_k g}$

D. $\frac{\omega_0^2 R^2}{18\mu_k g}$

E. $\frac{\omega_0^2 R^2}{20\mu_k g}$

Solution

$$f = Ma$$

$$I\alpha = -fR$$

$$\frac{f}{M}t = \left(\omega_0 - \frac{fR}{I}t\right)R$$

$$f\left(\frac{1}{M} + \frac{R^2}{I}\right)t = \omega_0 R$$

$$t = \frac{I}{I + MR^2}\frac{M}{f}\omega_0 R$$

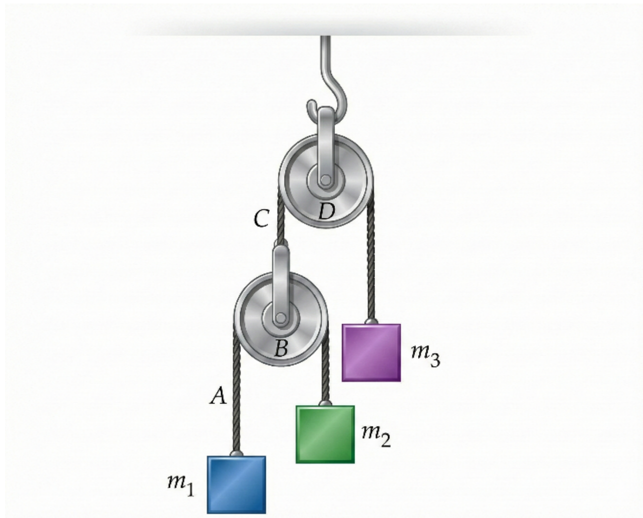
$$d = \frac{1}{2}at^2 = \frac{1}{2}\frac{f}{M}\left(\frac{I}{I + MR^2}\frac{M}{f}\omega_0 R\right)^2 = \frac{1}{2}\frac{M}{f}\left(\frac{1}{1 + MR^2/I}\right)^2 \omega_0^2 R^2$$

$$= \frac{1}{2}\frac{M}{\mu_k Mg}\left(\frac{1}{1+2}\right)^2 \omega_0^2 R^2 = \frac{\omega_0^2 R^2}{18\mu_k g}$$

11. The numbers in this question are exact.
In the figure below, masses m_1 and m_2 are connected by a light string *A* over a light, frictionless pulley *B*. The axle of pulley *B* is connected by a second light string *C* over a second light, frictionless pulley *D* to a mass m_3. Pulley *D* is suspended from the ceiling by an attachment to its axle. The system is released from rest. If m_1 = 1 kg, m_2 = 2 kg, m_3 = 3 kg, what is the acceleration of block m_1? Choose the positive direction to be upward.

此题中之数字皆为精确值。
在下图中，质量 m_1 和 m_2 通过一根轻绳 *A* 连接在一个轻的无摩擦滑轮 *B*

上。滑轮 B 的轴通过第二根轻绳 C 经过第二个轻的无摩擦滑轮 D 连接到质量 m_3。滑轮 D 以一个连接件固定悬挂在天花板上。系统从静止释放。如果 $m_1 = 1$ kg，$m_2 = 2$ kg，$m_3 = 3$ kg，则 m_1 的加速度是多少？取向上为正。



A. $\frac{2}{7}g$

B. $\frac{1}{3}g$

C. $\frac{7}{17}g$

D. $\frac{5}{12}g$

E. $\frac{6}{13}g$

$$T - m_2 g = m_2(a - A)$$

$$\frac{T}{m_2} - g = a - A$$

$$\frac{T}{m_1} - g = -a - A$$

$$\frac{T}{m_1} + \frac{T}{m_2} - 2g = -2A$$

$$T = 2\frac{m_1 m_2}{m_1 + m_2}(g - A)$$

$$2T - m_3 g = m_3 A$$

$$4\frac{m_1 m_2}{m_1 + m_2}(g - A) - m_3 g = m_3 A$$

$$\left(m_3 + \frac{4m_1 m_2}{m_1 + m_2}\right)A = \left(\frac{4m_1 m_2}{m_1 + m_2} - m_3\right)g$$

$$\big((m_1 + m_2)m_3 + 4m_1 m_2\big)A = (4m_1 m_2 - (m_1 + m_2)m_3)g$$

$$A = \frac{4m_1 m_2 - (m_1 + m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3}g$$

$$a = A + \frac{T}{m_2} - g = A + 2\frac{m_1}{m_1 + m_2}(g - A) - g$$

$$= \left(1 - \frac{2m_1}{m_1 + m_2}\right)(A - g)$$

$$= \left(1 - \frac{2m_1}{m_1 + m_2}\right)\left(\frac{4m_1 m_2 - (m_1 + m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3} - 1\right)g$$

$$= \frac{m_1 - m_2}{m_1 + m_2}\frac{2(m_1 + m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3}g$$

$$= \frac{2(m_1 - m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3}g$$

$$-a - A = -\frac{2(m_1 - m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3}g - \frac{4m_1 m_2 - (m_1 + m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3}g$$

$$= \frac{-2(m_1 - m_2)m_3 - 4m_1 m_2 + (m_1 + m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3}g$$

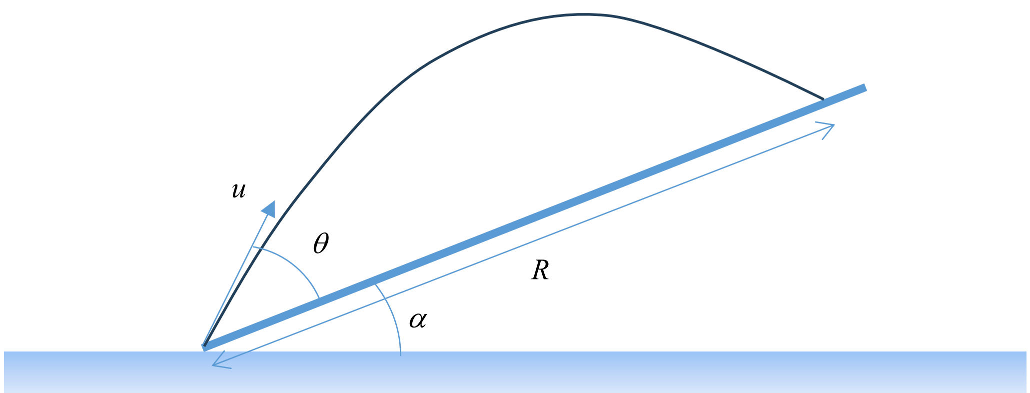
$$= \frac{-4m_1 m_2 - m_1 m_3 + 3m_2 m_3}{4m_1 m_2 + m_1 m_3 + m_2 m_3}g$$

Questions 12 and 13

An object on an inclined plane is launched with a speed u and at an angle θ measured from the upward direction along the plane, as shown in the figure below. The inclination angle of the plane is α, where $0^\circ < \alpha < 90^\circ$ and $0^\circ < \theta < 90^\circ - \alpha$. The object will hit the inclined plane at a point located a distance R up the plane.

问题 12 和 13

一个物体在倾斜面上以速率 u 被发射，发射角度从沿着倾斜面向上方向测量为 θ，如下图所示。倾斜面的倾角为 α，其中 $0° < \alpha < 90°$ 且 $0° < \theta < 90° - \alpha$。该物体将在倾斜面上距离为 R 的位置碰撞倾斜面。



12. Find the maximum value of R if α is fixed but θ may vary.

如果 α 固定但 θ 可变，找到 R 的最大值。

A. $\frac{u^2}{g}$

B. $\frac{u^2}{g}(1 - \sin\alpha)$

C. $\frac{u^2}{g\cos\alpha}(1 - \sin\alpha)$

D. $\frac{u^2}{g\cos^2\alpha}(1 - \sin\alpha)$

E. $\frac{u^2}{g\cos\alpha}$

13. If $\alpha = 20^\circ$, what is the angle θ that will lead to maximum R?

如果 $\alpha = 20°$，则 θ 为何值时 R 最大？

A. 10°

B. 25°

C. 35°

D. 40°

E. 45°

Solution

$$a_x = -g \sin \alpha$$

$$a_y = -g \cos \alpha$$

$$x = ut \cos \theta - \frac{1}{2} g t^2 \sin \alpha$$

$$y = ut \sin \theta - \frac{1}{2} g t^2 \cos \alpha$$

$$0 = uT \sin \theta - \frac{1}{2} g T^2 \cos \alpha$$

$$T = \frac{2u \sin \theta}{g \cos \alpha}$$

$$R = u \frac{2u \sin \theta}{g \cos \alpha} \cos \theta - \frac{1}{2} g \left(\frac{2u \sin \theta}{g \cos \alpha} \right)^2 \sin \alpha$$

$$= \frac{2u^2 \sin \theta \cos \theta}{g \cos \alpha} - \frac{2u^2 \sin^2 \theta}{g \cos^2 \alpha} \sin \alpha$$

$$= \frac{2u^2}{g \cos^2 \alpha} \sin \theta \left(\cos \alpha \cos \theta - \sin \alpha \sin \theta \right)$$

$$= \frac{2u^2}{g \cos^2 \alpha} \sin \theta \cos(\theta + \alpha)$$

$$= \frac{u^2}{g \cos^2 \alpha} [\sin(2\theta + \alpha) - \sin \alpha]$$

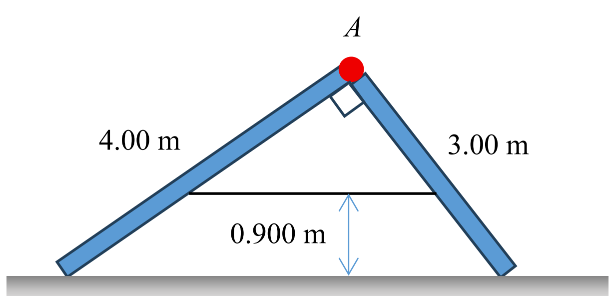
$$R_{\max} = \frac{u^2}{g \cos^2 \alpha} [1 - \sin \alpha]$$

$$2\theta + \alpha = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4} - \frac{\alpha}{2}$$

14. Two ladders, 4.00 m and 3.00 m long, are hinged at point *A* and tied together by a horizontal rope 0.900 m above the floor, as shown in the figure below. The ladders weigh 480 N and 360 N, respectively, and the center of mass of each is at its center. Assume that the floor is frictionless. If an 800-N painter stands at point *A*, find the tension in the horizontal rope.

两架梯子，分别长 4.00 米和 3.00 米，在 *A* 点铰接，并通过一根距地面 0.900 米的水平绳子绑在一起，如下图所示。这两架梯子的重量分别为 480 N 和 360 N，且每架梯子的质心位于其中点。假设地面是平滑的。如果一个重 800 N 的油漆工站在 *A* 点，求水平绳子的张力。



A. 937 N

B. 1010 N

C. 1160 N

D. 1250 N

E. 1380 N

Solution

Zero torque on the whole system about the left contact point with the ground:

$$5.00N_R = 2.00 \times \cos\left(\tan^{-1}\frac{3.00}{4.00}\right) \times 480 + \left(5.00 - 1.50\sin\left(\tan^{-1}\frac{3.00}{4.00}\right)\right) \times 360 + 4.00 \times \cos\left(\tan^{-1}\frac{3.00}{4.00}\right) \times 800$$

$$= 2.00 \times \frac{4}{5} \times 480 + \left(5.00 - 1.50 \times \frac{3}{5}\right) \times 360 + 4.00 \times \frac{4}{5} \times 800$$

$$= 768 + 1476 + 2560$$

$$N_R = 960.8 \text{ N}$$

Zero torque on the right ladder about A:

$$3.00 \times \sin\left(\tan^{-1}\frac{3.00}{4.00}\right) \times N_R$$
$$= 1.50 \times \sin\left(\tan^{-1}\frac{3.00}{4.00}\right) \times 360 + \left(3.00 \times \cos\left(\tan^{-1}\frac{3.00}{4.00}\right) - 0.90\right) T$$

$$3.00 \times \frac{3}{5} \times 960.8 = 1.50 \times \frac{3}{5} \times 360 + \left(3.00 \times \frac{4}{5} - 0.90\right) T$$

$$T = 2.00 \times \frac{3}{5} \times 960.8 - \frac{3}{5} \times 360 = 936.96$$

15. An astronaut inside a spacecraft is orbiting a black hole in circular path at a distance of 120 km from the blackhole. The black hole is 5.00 times the mass of the sun (solar mass = 2.00×10^{30} kg) and has a Schwarzschild radius of 15.0 km. Since the orbital radius is significantly larger than the Schwarzschild radius, one may assume that Newton's law of gravitation holds. The astronaut is positioned inside the spaceship such that one of her 0.030 kg ears is 6.0 cm farther from the black hole than the center of mass of the spacecraft and the other ear is 6.0 cm closer. What is the tension between her ears? Assume that orbits are circular. (Since her whole body orbits with the same angular velocity, one ear is moving too slowly for the radius of its orbit and the other is moving too fast. Hence her head must exert forces on her ears to keep them in their orbits.)

一名宇航员在宇宙飞船内正在以距黑洞中心 120 公里的圆形轨道绕黑洞飞行。该黑洞的质量为太阳质量的 5.00 倍（太阳质量 = 2.00×10^{30} kg），其史瓦西半径为 15.0 公里。由于轨道半径远大于史瓦西半径，可以假设牛顿引力定律成立。宇航员的一只质量为 0.030 kg 的耳朵距离黑洞比飞船的质心远 6.0 cm，而另一只耳朵则近 6.0 cm。她耳朵之间的张力是多少？。（由于她的整个身体以相同的角速度绕黑洞旋转，一只耳朵的运动速率对于其轨道半径来说太慢，而另一只耳朵的运动速率则太快。因此，她的头必须对她的耳朵施加力以保持它们在各自的轨道上。）

A. 700 N

B. 1400 N

C. 2100 N

D. 2800 N

E. 3500 N

Solution

$$m\omega^2 r = \frac{GMm}{r^2}$$

$$m\omega^2 (r+\delta) = \frac{GMm}{(r+\delta)^2} + T$$

$$m\omega^2 (r-\delta) = \frac{GMm}{(r-\delta)^2} - T$$

$$T = m\omega^2 (r+\delta) - \frac{GMm}{(r+\delta)^2} = m\omega^2 r \frac{r+\delta}{r} - \frac{GMm}{(r+\delta)^2}$$

$$= \frac{GMm}{r^2}\frac{r+\delta}{r} - \frac{GMm}{(r+\delta)^2} = 2084.374 \text{ N}$$

$$= \frac{GMm}{r^2}\left(1+\frac{\delta}{r}\right) - \frac{GMm}{r^2}\left(1-\frac{2\delta}{r}+\cdots\right) = \frac{GMm}{r^2}\frac{3\delta}{r} = 2084.375 \text{ N}$$