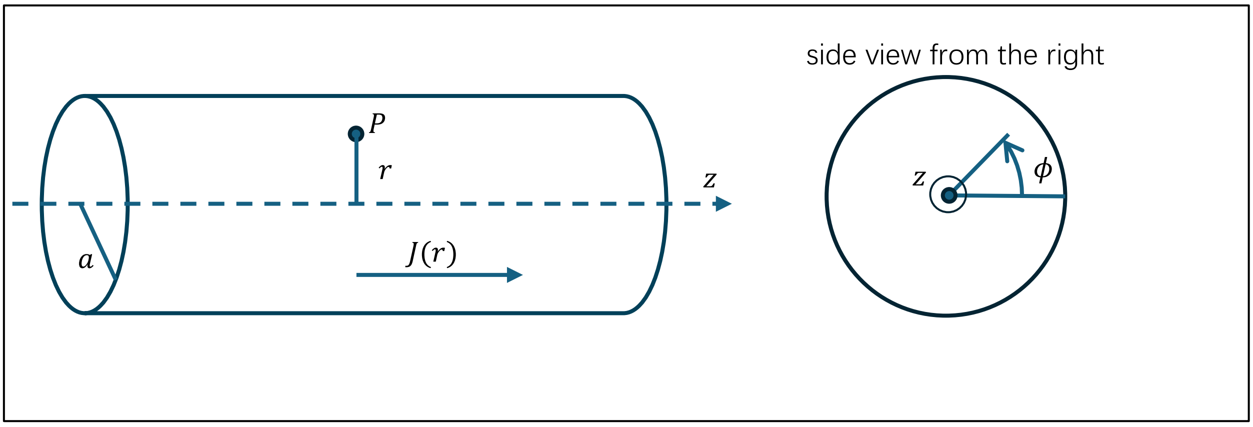
Question 1

In this problem we consider how current flows in conductors in direct current (DC) and alternating current (AC) settings.

An azimuthal symmetrical time independent current density $J(r) > 0$ flows through a long and straight cylindrical wire with radius a along $+z$ direction. We first consider a classical model of the current. In the lab frame, denoted as S, we imagine the electrons flow in opposite direction of the current with a uniform non-relativistic drift velocity $v \ll c$ towards $-z$ direction, among the fixed uniformly distributed positive charges. We assume the number density of electrons in the wire could vary with r and is denoted as $n_e(r)$ while the positive charges have a uniform number density, n_p. We also assume the wire is non-magnetic. Note: n_e and n_p are not charge densities.



(a) [1 pt] Write down the current density $J(r)$ in terms of electron charge e, $n_e(r), r$ and v. Take $+z$ as positive.

Solution:

$$J(r) = e\, n_e(r)\, v$$

(b) Consider a point P which is $r < a$ distance away from the z axis and where $J(r)$ is non-zero,

i. [1 pt] find the magnitude of the magnetic field, in terms of an integral involving $e, v, n_e(r), r$ and other universal constants and indicate its direction;

Solution: Since the wire is cylindrically symmetrical, Ampere's law gives:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int J(r) da$$

$$\Rightarrow B = B_\phi(r) = \frac{\mu_0 \int_0^r e\, n_e(r') v\, r' dr'}{r}$$

$$\vec{B}(r) = \frac{\mu_0 e v \int_0^r n_e(r') r' dr'}{r} \hat{\phi}$$

ii. [1 pt] find the magnitude of the electric field, in terms of an integral involving $e, v, n_p, n_e(r), r$ and other universal constants and indicate its direction.

Solution: Consider a gaussian surface of a cylinder with radius r and length L in the center of the wire. The total charge enclosed inside the surface is

$$Q_{encl} = \int_0^r e\left(n_p - n_e(r')\right)\, 2\pi r' dr' L$$

By symmetry the e-field is outward perpendicular to the z-axis. According to Gauss's law:

$$\vec{E}(r) = \frac{\left(\int_0^r e\left(n_p - n_e(r')\right)\, 2\pi r' dr' L\right)}{2\pi r\, L\, \epsilon_0} \hat{r}$$

$$= \frac{\int_0^r e\left(n_p - n_e(r')\right)\, r' dr'}{\epsilon_0\, r} \hat{r}$$

(c) [4 pt] When the current flows steadily, find the electron density $n_e(r)$ for $0 < r < a$ for a given v, in terms of n_p, e, v, a and other universal constants.

Solution: When the current is steady, the net force acting on the electrons at P must be zero.

Magnetic force on the electron

$$\vec{F_B} = -e\vec{v} \times \vec{B} = -evB(r)\left(-\hat{z} \times \hat{\phi}\right) = evB(r)(-\hat{r})$$

$$\vec{F}_B = \frac{\mu_0 e^2 v^2 \int_0^r n_e(r')r'dr'}{r}(-\hat{r})$$

Electric force:

$$\vec{F}_E = -eE(r)\hat{r}$$

Net force =0 requires:

$$\vec{F}_B + \vec{F}_E = 0$$

$$-\frac{\mu_0 e^2 v^2 \int_0^r n_e(r')r'dr'}{r} - e\,\frac{\int_0^r e\left(n_p - n_e(r')\right)\,r'dr'}{\epsilon_0\, r} = 0$$

$$\Rightarrow \mu_0\epsilon_0\, e^2 v^2 \int_0^r n_e(r')r'dr' + \int_0^r e^2\left(n_p - n_e(r')\right)\,r'dr' = 0$$

$$\Rightarrow \frac{v^2}{c^2}\int_0^r n_e(r')r'dr' + \int_0^r \left(n_p - n_e(r')\right)\,r'dr' = 0$$

$$\Rightarrow \left(1-\frac{v^2}{c^2}\right)\int_0^r n_e(r')r'dr' = \int_0^r n_p\ r'dr'$$

$$\Rightarrow \left(1-\frac{v^2}{c^2}\right)\int_0^r n_e(r')r'dr' = \frac{n_p\, r^2}{2} \qquad (1)$$

To yield r^2 dependence from the integral $\int_0^r n_e(r')r'dr'$,

$n_e(r)$ must be a constant in $0 < r' < r$.

Thus, $n_e(r) = n_e$
and from Eq. (1) we have

$$n_e = \frac{n_p}{1-\frac{v^2}{c^2}} \qquad (2)$$

Eq. (2) show the density of the electron is higher than that of the positive charge near the center of the wire. Since the total number of electron per unit length is the same as that of the positive charge, the electrons only occupy a cylinder with radius R such that:

$$n_e\,\pi R^2 = n_p \pi a^2$$

$$\Rightarrow R = a\sqrt{1-\frac{v^2}{c^2}}$$

For $r \leq R, n_e(r) = \frac{n_p}{1-\frac{v^2}{c^2}}$

For $a \geq r > R,\ n_e(r) = 0$

$$n_e(r) = \begin{cases} \dfrac{n_p}{1-\dfrac{v^2}{c^2}} & r \leq R \\ 0 & R < r \leq a \end{cases}$$

(d) [1 pt] Sketch net charge distribution along the radial direction outward for $0 < r < a$.

Solution: draw a step function like form of $n_e(r)$

(e) [2 pt] Find the voltage difference between surface of the wire and the center of the wire, in terms of electron charge e, n_p , a and v. The (surface / center) has a higher voltage. (circle the correct answer.)

Solution: As the negative charges move to the center, the surface of the wire is positively charged and it has a higher potential. ➔ Surface

$$\Delta V = -\int_0^a E(r)dr = -\int_0^R \frac{e\,(n_p - n_e)\,r}{2\epsilon_0} dr - \int_R^a \frac{en_p r}{2\epsilon_0} - \frac{en_e R^2}{2\epsilon_0 r} dr$$

$$= -en_p\left(1 - \frac{1}{1 - \frac{v^2}{c^2}}\right)\frac{R^2}{4\,\epsilon_0} - \frac{en_p}{4\epsilon_0}(a^2 - R^2) + \frac{en_p a^2}{2\epsilon_0}\ln\frac{a}{R}$$

$$= e\,n_p\left(\frac{v^2}{c^2}\right)\frac{a^2}{4\epsilon_0} - e\,n_p\left(\frac{v^2}{c^2}\right)\frac{a^2}{4\epsilon_0} + e\frac{n_p a^2}{2\epsilon_0}\ln\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= e\frac{n_p a^2}{2\epsilon_0}\ln\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now let’s look at the current in a reference frame which moves as the same velocity as the electrons do, denote the reference frame as S'. Therefore, in the S' frame, the electrons are stationary and the positive charge and the wire is moving at v towards $+z$ direction.

(f) [1 pt] In the S' frame, calculate the net force acting on the electron at P, in terms of electron charge e, n_p and v.

Solution: the force at P is only due to electric field on the electron in S' frame only as the electron is at rest.

$$F_B = 0$$

$$F_{net} = \frac{-e^2(n_p - n_e)r}{2\epsilon_0}$$

$$= \frac{\frac{e^2\ n_p r}{2\epsilon_0}\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} > 0\ (outward)$$

There is a discrepancy between the motion of the electrons in S and S' ! Should the electrons flow steady or should they subject to non-zero net force at P? This discrepancy will become larger when v increases, too.

Suppose in the future we discover a new material in which the drift velocity can be comparable to the speed of light. Now we need to consider the relativistic effect. Let's hold on to the observation that when the current is flowing steadily, the wire is overall neutral in the S frame.

(g) [1 pt] The number density of the positive charge in S frame and that in S' frame are denoted as n_p and n_p', respectively. Find the relation between n_p and n_p' in terms of v and c.

Solution: Due to length contraction, the density of the particle increases as

$n_p' = \gamma\, n_p$, where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

(h) [2 pt] Explicitly show in the S' frame, the net force on the electrons at P become zero with consideration of relativistic effect.

Solution:

E-field by the positive charges: (outward)

$$E_+ = \frac{en'_p \pi r^2 L}{\epsilon_0\, 2\pi r\, L} = \frac{en'_p\, r}{2\epsilon_0} = \frac{e\,\gamma\, n_p\, r}{2\epsilon_0}$$

E-field by the negative charges: (inward)

$$E_- = \frac{en'_e \pi r^2 L}{\epsilon_0\, 2\pi r\, L} = \frac{en'_e\, r}{2\epsilon_0} = \frac{e\left(\frac{n_e}{\gamma}\right)\, r}{2\epsilon_0} = \frac{e\, r}{2\epsilon_0}\frac{n_p}{1-\frac{v^2}{c^2}}\frac{1}{\gamma} = \frac{e\, r}{2\epsilon_0} n_p\, \gamma^2 \frac{1}{\gamma} = \frac{e\, r}{2\epsilon_0} n_p\, \gamma = E_+$$

Therefore, the electric field is zero. The electron is not moving in S' so there is no magnetic force acting on them. So the net force acting on the electrons at P is zero. That is consistent with the picture in S frame.

Partial Credit:

For the electron:

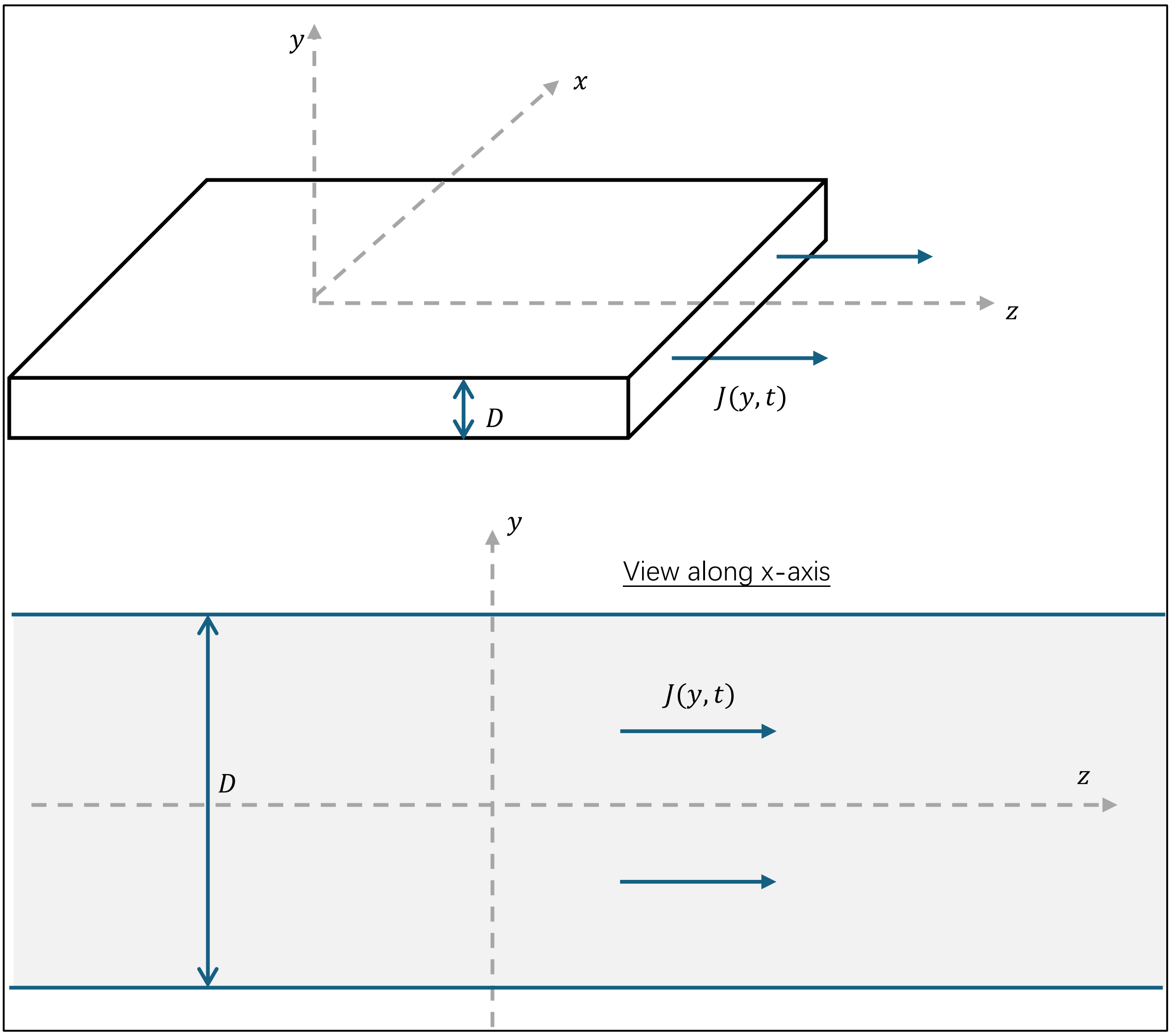
$$(rest\ in\ S')\ \ \gamma n'_e = n_e\ (moving\ in\ S)$$

Use the density of the electron in term of n_p in part (c)

$$n_e = \frac{n_p}{1-\frac{v^2}{c^2}}$$

For conductors in reality, all of them have a very small drift velocity, thus, relativistic effect can actually be ignored. In addition, the magnetic force on the electrons is also negligible. Therefore, in DC, we can assume the current density is uniform throughout the cross-section of the conductor.

In the following, we will consider a conductor of an infinitely large (along x and z) slab with width D as shown below. The z-axis is in the middle of the slab. The conducting slab obeys Ohm's Law $\vec{J} = \sigma\vec{E}$. with conductivity $\sigma = 3.77 \times 10^{7}\ \Omega^{-1}m^{-1}$. The dielectric constant of the slab is $K = 1$. We can ignore the relativistic effect from here on.



To warm up, we consider a uniform current density with increasing magnitude as

$J(y,t) = j_0 + \alpha\, t$, where j_0 and α are positive constants and $j_0 \gg \frac{\alpha\epsilon_0}{\sigma}$.

(i) [2 pt] Find the magnetic field due to the current as a function of y and t in the conductor.

Solution: Consider a rectangular loop with dimension (y*w) on the x-y plane. According to Ampere's Law:

$$2Bw = \mu_0(j_0 + \alpha t)2yw + \frac{\mu_0\epsilon_0}{\sigma}\alpha\,(2yw)$$

$$\approx \mu_0(j_0 + \alpha t)2yw$$

$$\vec{B} = -\mu_0(j_0 + \alpha t)y\,\hat{x}$$

Partial credit:

Using Ampere's Law to set up equation for B:

Ignoring displacement current: 0.5pt

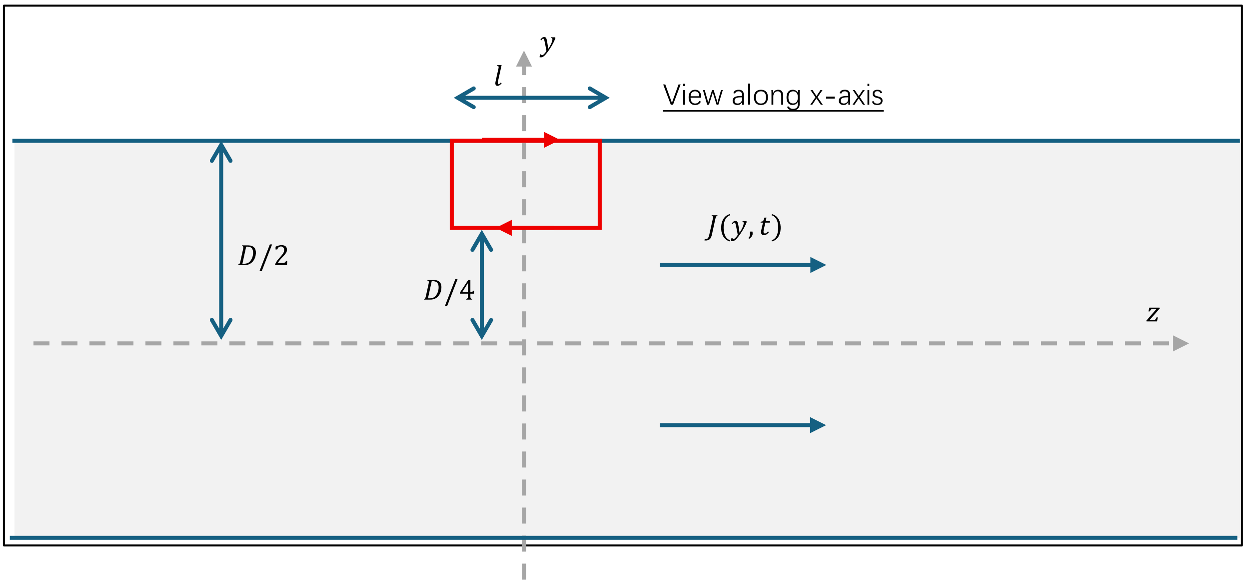
Wrong direction: - 0.5 pt

(j) [2 pt] Find the difference in the z-component of the electric field $\Delta E_z = E_z\left(y = \frac{D}{2}\right) - E_z\left(y = \frac{D}{4}\right)$ at any time, taking $E > 0$ as $\vec{E}$ points towards the $+z$ direction.

Solution: From Faraday's Law, we have (along a rectangular loop on y-z plane)

$$\left[E_z\left(\frac{D}{2}\right) - E_z\left(\frac{D}{4}\right)\right] l = -\frac{\partial}{\partial t}\int_{\frac{D}{4}}^{\frac{D}{2}} -\mu_0(j_0 + \alpha t)y\,l\,dy \qquad 1pt$$

$$\Delta E = \frac{\mu_0\alpha}{2}\left(\frac{D^2}{4} - \frac{D^2}{16}\right) = \frac{3\mu_0\alpha D^2}{32} \qquad 1pt$$



Partial Credit:

Use Faraday's Law: 1 pt

Wrong sign: -0.5pt

According to the Ohm's law, this yield a difference in the current density along y and this is referred to as skin effect that, for time varying current, the current density concentrates at the surface of the conductor. This effect occurs commonly in AC power transmission wire.

Now we consider an alternating current (AC) with frequency $f = 50$Hz passing through the slab. The current density flows along z-axis is $J(y,t) = Re\left(j(y)e^{i\omega t}\right)$.

(k) [1 pt] Show that for the given parameters, the displacement current is much smaller than the free current.

(l) [2 pt] Derive a differential equation for $j(y)$ by ignoring the displacement current.

(m) [4 pt] In the limit the slab is very thick $D \gg \sqrt{1/\sigma\mu_0\omega}$ and we are interested in the current near the bottom edge $\left(y + \frac{D}{2}\right) \ll \sqrt{1/\sigma\mu_0\omega}$, solve for the current density $j(y)$ for $j\left(\frac{D}{2}\right) = j\left(-\frac{D}{2}\right) = j_0 = 1\ \mathrm{Am^{-2}}$. Express $j(y)$ in terms of $\sigma, \epsilon_0, \mu_0, \omega, D$ and j_0

(n) [2 pt] Write down $J(y,t)$ and sketch the current distribution $J(y,0)$ for $-\frac{D}{2} \leq y \leq 0$ for $D = 10\ \mathrm{cm}$ and the given parameters.

Suppose the frequency of the AC is very high such that the displacement current is much larger than the free current.

(o) [3 pt] Show that a standing wave of $j(y)$ will be formed in the slab along y and find the wavelength of the standing wave in terms of ω.

Solution:

Let $E = Re(E_z e^{i\omega t})$

(k)

$$J_F = \sigma E_z\ , J_D = \epsilon_0 i\omega E_z$$

$$\frac{\sigma}{\epsilon_0 \omega} = \frac{3.77 \times 10^7}{8.85 \times 10^{-12} * 2\pi(50)} = 1.36 \times 10^{16} \gg 1$$

(l) Apply Ampere's Law:

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial}{\partial t} E = \mu_0 \sigma E_z(y) e^{i\omega t} + i\omega \mu_0 \epsilon_0 E_{z(y)} e^{i\omega t}$$
$$\approx \mu_0 \sigma E_z(y) e^{i\omega t}$$

could use J instead of E here as $J = \sigma E$

$$\frac{\partial}{\partial \mathrm{t}} \nabla \times B = \nabla \times (-\nabla \times E) = i\omega \mu_0 \sigma E_z(y) e^{i\omega t}$$

$$-\nabla \times \left(\nabla \times E_z(y) e^{i\omega t}\right) = i\omega \mu_0 \sigma E_z(y) e^{i\omega t}$$

$$-\nabla\left(\nabla \cdot E_z(y) e^{i\omega t}\right) + \nabla^2 E_z(y) e^{i\omega t} = i\omega \mu_0 \sigma E_z(y) e^{i\omega t}$$

$$\nabla^2 E_z(y) e^{i\omega t} = i\omega \mu_0 \sigma E_z(y) e^{i\omega t}$$

$$\nabla^2 E_z(y) = i\omega \mu_0 \sigma E_z(y)$$

$$\frac{\partial^2}{\partial y^2} j(y) = i\omega \mu_0 \sigma\, j(y)$$

Partial Credit:

Use Ampere's Law

Use Ohm's law: $J = \sigma E$

Notice that $\nabla \cdot E = 0$ since $\vec{E} = E_z(y)\hat{z}$

(m) Let $k^2 = i\omega\mu_0\sigma$ and $k = \pm\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)$. Solution of the differential equation:

$$j(y) = Ae^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)y} + Be^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)y}$$

For $y = D/2$ and $y = -D/2$, $j = j_0$

$$A = B$$

$$j_0 = A\left(e^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\frac{D}{2}} + e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\frac{D}{2}}\right)$$

$$j(y) = \frac{j_0\left(e^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)y} + e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)y}\right)}{\left(e^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\frac{D}{2}} + e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\frac{D}{2}}\right)}$$

For $D \gg \sqrt{\frac{1}{\sigma\mu_0\omega}}$, $e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\frac{D}{2}} \approx 0$

$$j(y) = \frac{j_0\left(e^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)y} + e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)y}\right)}{e^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\frac{D}{2}}}$$

$$= j_0\left(e^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\left(y-\frac{D}{2}\right)} + e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\left(y+\frac{D}{2}\right)}\right)$$

For $\left(y + \frac{D}{2}\right) \ll \sqrt{\frac{1}{\sigma\mu_0\omega}}$, we have

$$j(y) = j_0\left(e^{\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\left(y+\frac{D}{2}-D\right)} + e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\left(y+\frac{D}{2}\right)}\right)$$

$$j(y) \rightarrow j_0 e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)\left(y+\frac{D}{2}\right)}$$

(n) $J(y,t) = j_0 e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}\left(y+\frac{D}{2}\right)}\cos\left(\sqrt{\frac{\omega\mu_0\sigma}{2}}\left(y+\frac{D}{2}\right)-\omega t\right)$

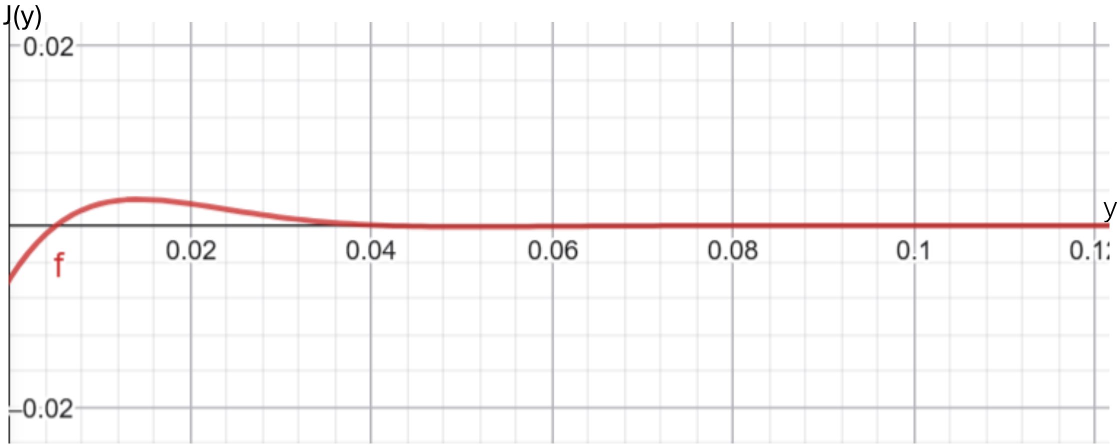
$$J(y,0) = j_0 e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}\left(y+\frac{D}{2}\right)}\cos\left(\sqrt{\frac{\omega\mu_0\sigma}{2}}\left(y+\frac{D}{2}\right)\right)$$

The function decays with a depth $\sqrt{\left(\frac{2}{\omega\mu_0\sigma}\right)} = 0.012\ m$. It also oscillates with a wavelength $2\pi\sqrt{\left(\frac{2}{\omega\mu_0\sigma}\right)} = 0.07\ m$

Sketch:

quick decay to zero

oscillation along



(o)

$$\nabla \times B = \mu_0\epsilon_0\frac{\partial}{\partial t}E = i\omega\mu_0\epsilon_0 E_{z(y)}e^{i\omega t}$$

$$\nabla \times (-\nabla \times E) = \frac{\partial}{\partial \mathrm{t}} \nabla \times B = -\omega^2 \mu_0 \epsilon_0 E_z(y) e^{i\omega t}$$

$$-\nabla \times \left(\nabla \times E_z(y) e^{i\omega t}\right) = -\frac{\omega^2}{c^2 E_z(y) e^{i\omega t}}$$

$$-\nabla\left(\nabla \cdot E_z(y) e^{i\omega t}\right) + \nabla^2 E_z(y) e^{i\omega t} = -\frac{\omega^2}{c^2} E_z(y) e^{i\omega t}$$

$$\nabla^2 E_z(y) e^{i\omega t} = -\frac{\omega^2}{c^2 (y) e^{i\omega t}}$$

$$\nabla^2 E_z(y) = -\frac{\omega^2}{c^2 E_z(y)}$$

$$\frac{\partial^2}{\partial y^2} j(y) = -\frac{\omega^2}{c^2} j(y)$$

Solution of $j(y)$ is

$$j(y) = A e^{i\frac{\omega}{c}y} + B e^{-i\frac{\omega}{c}y}$$

With the boundary condition $j\left(\frac{D}{2}\right) = j\left(-\frac{D}{2}\right)$:

$$A = B$$

Thus,

$$J(y,t) = Re\left(A\left(e^{i\left(\frac{\omega}{c}y+\omega t\right)} + e^{-i\left(\frac{\omega}{c}y-\omega t\right)}\right)\right)$$

$$= A\cos\left(\frac{\omega}{c}y + \omega t\right) + A\cos\left(\frac{\omega}{c}y - \omega t\right)$$

$$= 2A\cos\left(\frac{\omega}{c}y\right)\cos\omega t$$

It is a standing wave solution.

Wavelength =

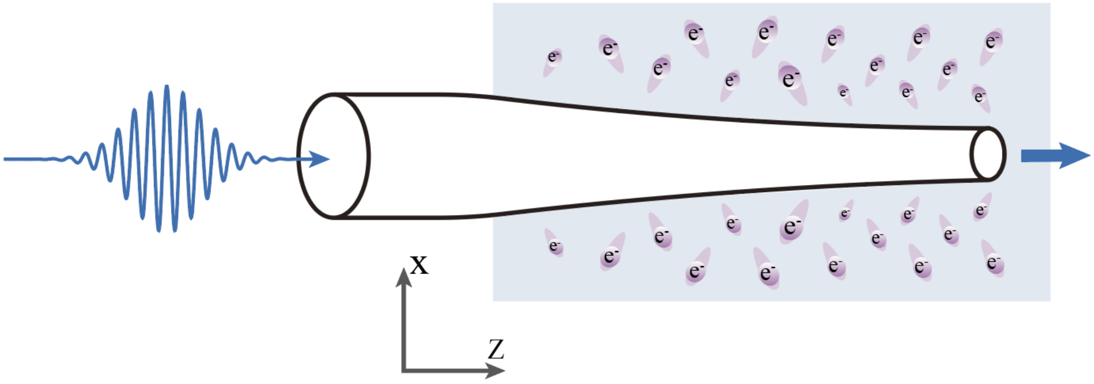
$$\lambda = \frac{2\pi c}{\omega}$$

Question 2:

在强场物理中有一个颇为有趣的现象：强激光脉冲能够电离空气并激发**空气等离子体波**。光脉冲的 **Ponderomotive Force(有质动力)** 会额外作用在自/由电子上(正离子可以近似认为静止)，产生新的分布，从而对波的传播产生显著影响。考虑一个简化的一维模型：在沿 x方向分布的一薄层空气等离子体中，电子数密度记为 $n_e(x,t)$。一束载频为 ω的线偏振激光以适当方式入射，激发出沿 z方向传播的空气等离子体波。在空气中的某一局部区域内，该等离子体波在横向 （x方向）上的电场分布可以近似为一高斯分布：

$$\mathbf{E}(x,t) = \hat{x} E_0\, e^{-x^2/L^2} \cos(\omega t),$$

其中 E_0为电场振幅峰值，L 为该模式在空气中横向分布的特征宽度。可假设电子速度处于非相对论区间 $v \ll c$，并忽略碰撞效应。



(a) 考虑一电子，其电荷为 $-e$、质量为 m_e，初始速度与初始位置均为 0。处于快速振荡电场 $\mathbf{E}(x,t)$中时，定义其有质动力势 $U_p(x)$为该电子在驱动振荡过程中的周期平均动能，此定义在电场的纵向（z 方向）分布的包络随时间变化远慢于光学周期的极限下成立。请写出该电子在给定电场中的运动方程，并由此推导出 $U_p(x)$关于 x、E_0、L 及相关物理常数的显式表达式。

(b) 在有质动力势存在的情况下，电子会感受到额外的有质动力。请写出有质动力$F_{p,x}(x)$在 x方向上的分量 $F_{p,x}(x)$，并说明这一有质动力的方向。定性画出 $U_p(x)$随 x的分布曲线。并用一到两句话简要说明理由。

(c) 假设在包络的慢时间尺度上，电子可以被视为温度为T_e的热流体，并在有质动力与沿 x 方向的压力梯度之间达到准静态平衡。忽略重力和离子运动。其中电子压力为$\boldsymbol{p_e} = \boldsymbol{n_e k_B T_e}$。利用慢时间尺度下的受力平衡条件：

$$-\frac{\partial p_e}{\partial x} + n_e(x)\, F_{p,x}(x) = 0,$$

推导出 $n_e(x)$与 $U_p(x)$之间的微分方程，并求解平衡时电子密度$n_e(x)$。

(d) 利用 (a) 部分得到的 $U_p(x)$，将 $n_e(x)$显式写成关于 x、E_0、L、T_e 及物理常数

的函数。并讨论电子空穴区（即 n_e显著降低的区域）的宽度如何依赖于 L和 E_0。

(e) 在实际情况下，随着电子的迁移，局域的介电环境会随之改变，从而反过来影响光场的空间分布。根据(c)中得到的电子密度分布 $n_e(x)$，可得到相应的局域等离子体频率 $\omega_p^2(x)=\frac{n_e(x)e^2}{\varepsilon_0 m_e}$，请（i）建立微观模型，并推导出相对介电常数的表达式（不考虑碰撞，散射引起的耗散）；（ii）激光束会发生自聚焦还是自发散行为，请给出定性解释，（iii）假设有光场作用下的等离子体状态不随 z 坐标（纵向）持续发生变化，请找到自聚焦或自发散行为，对应焦距的近似表达式。

In strong-field physics, there is an intriguing phenomenon: an intense laser pulse can ionize air and excite an air plasma wave. The ponderomotive force associated with the pulse will exert on free electrons (positive ions can be viewed as stationary), thereby affecting propagation of the laser pulse itself. We consider a simplified one-dimensional model: a thick air plasma slab extends along the x-direction, with electron number density labeled as $n_e(x,t)$. A linear-polarized laser with carrier frequency ω is incident in such a way that it first creates air plasma and then propagates in it along the z-direction. The electric field distribution the laser field in the transverse direction (x-direction) can be approximated as a Gaussian profile:

$$\mathbf{E}(x,t)=\hat{x}E_0 e^{-x^2/L^2}\cos(\omega t),$$

where $E_0$is the peak amplitude of laser pulse and L is the characteristic width of the mode in the transverse direction in air. We assume that the electrons are nonrelativistic, $v \ll c$, and that electron-electron collision and electron-ion collision can be neglected.

(a) Consider an electron with charge $-e$, mass m_e and zero initial velocity and position. In a rapidly oscillating electric field $\mathbf{E}(x,t)$, the ponderomotive potential $U_p(x)$ is defined as the cycle-averaged kinetic energy of the driven oscillatory motion of the electron. This definition is valid in the limit where the field envelope in the propagation direction (z axis) varies much more slowly than the optical field. Write down the equation of motion for the electron in the given field, and use it to derive an explicit expression for $U_p(x)$ in terms of x, E_0, L, and the relevant physical constants.

(b) In the presence of the ponderomotive potential, the electrons will experience an effective force, the ponderomotive force. Find the x-component $F_{p,x}(x)$ of this force and define its direction. Sketch qualitatively the spatial profile of $U_p(x)$. Explain your reasoning briefly in one or two sentences.

(c) Assume that on the slow time scale far longer the period of the optical field, the electrons can be treated as a hot fluid, to which an electron temperature T_e can be assigned. And those electrons may strike a quasi-equilibrium due to the coexisting ponderomotive force and the thermal pressure along the x-direction. Neglecting gravity and the motion of the ion. The electron pressure is expressed

$$p_e = n_e k_B T_e.$$

On the slow time scale, the force balance condition reads

$$-\frac{\partial p_e}{\partial x} + n_e(x)\, F_{p,x}(x) = 0$$

Derive the differential equation relating $n_e(x)$ to $U_p(x)$, and solve for the equilibrium electron density distribution $n_e(x)$, using n_0 as the reference electron density as $x \to \infty$.

(d) Using the expression for $U_p(x)$ obtained in part (a), find the explicit form of $n_e(x)$ in terms of x, E_0, L, T_e and other necessary constants. Discuss how the width of the electron depletion region (i.e., the region where n_e is significantly reduced) scales with L and E_0.

(e) In practice, as electrons redistribute, the local dielectric environment will vary as well. Such change will, in turn, affect the spatial profile and propagation of the laser field. Using the electron density distribution $n_e(x)$obtained in part (c), (i) please construction microscopic model to derive the spatial dependent dielectric function; (ii) please determin is the laser beam will be converging or divering when propagating inside the plasma.; (iii), suppose the light-driven redistrubtion of electron density has negligible z-dependence. Please find the effective focal length of this focusing (defucosing) "device".

标准解答

(a) Time-averaged Ponderomotive Potential

(a)　有质动力势的时间平均表达式

电子电荷 $-e$、质量 m_e，一维快速振荡方程：

$$m_e \ddot{x}_{\mathrm{osc}} = -eE(x,t).$$

带入初始条件$x_{\mathrm{osc}}(0) = 0$与$\dot{x}_{\mathrm{osc}}(0)$=0 ：

$$x_{\text{osc}}(t) = \frac{eE(x)}{m_e\omega^2}(\cos(\omega t) - 1) \qquad \left(E(x) = E_0 e^{-\frac{x^2}{L^2}}\right) \qquad .$$

瞬时速度：

$$\dot{x}_{\text{osc}}(t) = -\frac{eE(x)}{m_e\omega}\sin(\omega t).$$

快振荡动能：

$$K(t) = \frac{1}{2}m_e\dot{x}_{\text{osc}}^2 = \frac{1}{2}m_e\left(\frac{eE(x)}{m_e\omega}\right)^2\sin^2\omega t$$

对若干光学周期取时间平均：

$$\langle K\rangle_t = \frac{e^2E^2(x)}{2m_e\omega^2}\langle\sin^2\omega t\rangle_t = \frac{e^2E^2(x)}{2m_e\omega^2}\cdot\frac{1}{2} = \frac{e^2}{4m_e\omega^2}E^2(x).$$

因此

$$U_p(x) = \langle K\rangle_t = \frac{e^2}{4m_e\omega^2}E^2(x).$$

(b) Ponderomotive Force and Effective Potential Well

(b)　振荡压力与等离子体“势阱”

(bi) 计算 $\boldsymbol{F_{p,x}(x)}$及方向

$$\mathbf{F}_p(x) = -\nabla U_p(x) \Rightarrow F_{p,x} = -\frac{dU_p}{dx}.$$

由

$$U_p(x) = U_0 e^{-\frac{2x^2}{L^2}}, U_0 \equiv \frac{e^2E_0^2}{4m_e\omega^2},$$

求导：

$$\frac{dU_p}{dx} = U_0 e^{-2x^2/L^2}\left(-\frac{4x}{L^2}\right) = -\frac{4x}{L^2}U_0 e^{-2x^2/L^2}.$$

因此

$$F_{p,x}(x) = -\frac{dU_p}{dx} = +\frac{4x}{L^2}U_0 e^{-\frac{2x^2}{L^2}} = \frac{e^2E_0^2}{m_e\omega^2}\frac{x}{L^2}e^{-\frac{2x^2}{L^2}}.$$

符号分析：

- 当 $x > 0$时，$F_{p,x} > 0$：力指向正 x，把电子往远离 $x = 0$推；
- 当 $x < 0$时，$F_{p,x} < 0$：力指向负 x，同样把电子推离 $x = 0$的高场区。

结论：
振荡压力总是把电子从高强度中心（$x = 0$ 附近）**向外推开**。

$U_p(x)$在 $x = 0$处为最大值：

$$U_p(0) = U_0, U_p(|\, x\,| \to \infty) \to 0.$$

因此 $x = 0$是一个**势垒顶**，而不是势阱。对电子来说，靠近 $x = 0$需要额外能量，故高强电场区域形成了一个**势垒**，电子被排斥到两侧。

- 能画出或描述 $U_p(x)$在 $x = 0$最高、两侧降低的形状

(c) Modification of Local Electron Density

(c) [3] 振荡压导致的局域电子密度调制

(ci) 推出 Boltzmann 型分布[3]

给：

$$p_e = n_e k_B T_e, -\frac{\partial p_e}{\partial x} + n_e F_{p,x} = 0.$$

代入：

$$-\frac{d}{dx}(n_e k_B T_e) + n_e(-\frac{dU_p}{dx}) = 0.$$

注意 $F_{p,x} = -\frac{dU_p}{dx}$（对带负电的电子，已经通过 U_p定义考虑了电荷号）。于是：

$$-k_B T_e \frac{dn_e}{dx} - n_e \frac{dU_p}{dx} = 0.$$

移项：

$$k_B T_e \frac{dn_e}{dx} + n_e \frac{dU_p}{dx} = 0 \Rightarrow \frac{1}{n_e}\frac{dn_e}{dx} = -\frac{1}{k_B T_e}\frac{dU_p}{dx}.$$

积分：

$$\int \frac{1}{n_e} dn_e = -\frac{1}{k_B T_e}\int dU_p \Rightarrow \ln n_e = -\frac{U_p}{k_B T_e} + C.$$

指数化：

$$n_e(x) = n_0 \exp\left(-\frac{U_p(x)}{k_B T_e}\right),$$

其中 $n_0 = e^C$为常数参考密度。

(d) 显式 $\boldsymbol{n_e(x)}$及耗尽宽度

由 (a)：

$$U_p(x) = U_0 e^{-2x^2/L^2}, U_0 = \frac{e^2 E_0^2}{4 m_e \omega^2}.$$

带入：

$$n_e(x) = n_0 \exp\left(-\frac{U_p(x)}{k_B T_e}\right) = n_0 \exp\left(-\frac{U_0}{k T_e} e^{-2x^2/L^2}\right),$$

在 $x = 0$处：

$$n_e(0) = n_0 \exp\left(-\frac{U_0}{k_B T_e}\right),$$

显著小于 n_0 当 $U_0 \gtrsim k_B T_e$。

关于宽度：

- $U_p(x)$的空间变化由高斯宽度 L控制，因而密度调制的特征尺度 $\Delta x \sim L$。
- 增大 E_0增加 U_0，使 $U_p(0)$相对 k_BT_e更大：
  - 中心处耗尽程度更强（$n_e(0)/n_0$ 更小）
  - 有效“空穴区”在 $|U_p(x)| \gtrsim k_BT_e$的范围内，其边界由 $\frac{U_0}{k_BT_e}e^{-2x^2/L^2} \sim 1$决定，
    近似有

$$x_{\text{depl}} \sim \frac{L}{\sqrt{2}}\sqrt{\ln\left(\frac{U_0}{k_BT_e}\right)}\ .$$

  - 所以增大 E_0会**略微扩大**耗尽区的宽度，并显著加深耗尽程度。

(e) 激光自聚焦效应

(i) 推导相对介电常数：

已有

$$U_p(x) = U_p(0)\exp\left(-\frac{2x^2}{L^2}\right), U_p(0) = \frac{e^2E_0^2}{4m_e\omega^2}.$$

于是

$$n_e(x) = n_0 exp\left[-\frac{U_p(0)}{k_BT_e}exp(-\frac{2x^2}{L^2})\right].$$

令

$$\alpha \equiv \frac{U_p(0)}{k_BT_e} \ll 1,$$

则

$$n_e(x) \approx n_0\left[1 - \alpha exp\left(-\frac{2x^2}{L^2}\right)\right].$$

因此

$$\omega_p^2(x)=\frac{n_e(x)e^2}{\varepsilon_0 m_e}\approx\omega_{p0}^2\left[1-\alpha exp\left(-\frac{2x^2}{L^2}\right)\right],$$

其中 $\omega_{p0}^2\equiv\frac{n_0e^2}{\varepsilon_0 m_e}$。

相对介电常数：

$$\varepsilon_r(x)\approx 1-\frac{\omega_{p0}^2}{\omega^2}+\frac{\omega_{p0}^2}{\omega^2}\ \alpha exp\left(-\frac{2x^2}{L^2}\right).$$

记

$$\varepsilon_r(\infty)=1-\frac{\omega_{p0}^2}{\omega^2},$$

则

$$\varepsilon_r(x)\approx\varepsilon_r(\infty)+\frac{\omega_{p0}^2}{\omega^2}\ \alpha exp\left(-\frac{2x^2}{L^2}\right).$$

(ii) 判断激光束的行为：

根据所得到的介电函数表达式，相对介电常数相对于平衡态$\varepsilon_r(\infty)$的改变量呈现为高斯分布。由此，等效折射率 $n=\sqrt{\varepsilon_r}$ 也表现出中心区域折射率较高、并向四周逐渐衰减至平衡态折射率 $n(\infty)=\sqrt{\varepsilon_r(\infty)}$ 的空间分布。这种分布等效于一个凸透镜结构，因此激光束在传播过程中会表现出自聚焦行为.　。

(iii) 计算等效焦距：

对于折射率

$$n(x)=\sqrt{\varepsilon_r(x)}=\sqrt{\varepsilon_r(\infty)+\frac{\omega_{p0}^2}{\omega^2}\ \alpha exp(-\frac{2x^2}{L^2})}=\sqrt{\varepsilon_r(\infty)}\times\sqrt{1+\frac{\Delta\varepsilon_r(x)}{\varepsilon_r(\infty)}},$$

其中$\Delta\varepsilon_r(x)=\frac{\omega_{p0}^2}{\omega^2}\ \alpha exp(-\frac{2x^2}{L^2})$

考虑$\Delta\varepsilon_r(x)$变化量相对于$\varepsilon_r(\infty)$很小，即$\Delta\varepsilon_r(x)\ll\varepsilon_r(\infty)$，并对$n(x)$展开到一阶即

$$n(x) \approx \sqrt{\varepsilon_r(\infty)} \times \left(1+\frac{1}{2}\frac{\Delta\varepsilon_r(x)}{\varepsilon_r(\infty)}\right).$$

同时考虑到$\Delta\varepsilon_r(x)=\frac{\omega_{p0}^2}{\omega^2}\alpha exp\left(-\frac{2x^2}{L^2}\right)\approx\alpha\frac{\omega_{p0}^2}{\omega^2}(1-\frac{2x^2}{L^2})$

故$n(x)$可以进一步化简为

$$n(x)=n_{eff}+\beta x^2.$$

其中n_{eff}和β分别为

$$n_{eff}=\sqrt{\varepsilon_r(\infty)}+\frac{1}{2}\alpha\frac{\omega_{p0}^2}{\omega^2}\frac{1}{\sqrt{\varepsilon_r(\infty)}},$$

$$\beta=-\alpha\frac{\omega_{p0}^2}{\omega^2}\frac{1}{\sqrt{\varepsilon_r(\infty)}L^2}$$

考虑等离子体中相互作用的有效长度$\boldsymbol{d}$，光场经过这段等离子体中相位改变的量为

$$\phi(x)=k_0n(x)\boldsymbol{d}=k_0\left(n_{eff}+\beta x^2\right)\boldsymbol{d}$$

其中$k_0n_{eff}\boldsymbol{d}$为固定相位，则相对相位变化由x^2项决定，根据傅里叶光学的中薄透镜成像中相因子公式

$$\phi_{lens}(x)=-\frac{k_0x^2}{2f}$$

其中$\boldsymbol{f}$为等效焦距，对比两式可以计算出空气等离子体自聚焦效应的等效焦距

$$\boldsymbol{f}=\frac{\omega^2\sqrt{\varepsilon_r(\infty)}L^2}{\boldsymbol{2}\alpha\omega_{p0}^2\boldsymbol{d}}$$

$$\boldsymbol{f}=\frac{\boldsymbol{\omega^2L^2}}{\boldsymbol{2\alpha d}}\left(\frac{\boldsymbol{\epsilon_0m_e}}{\boldsymbol{n_0e^2}}\right)\sqrt{1-\frac{n_0e^2}{\varepsilon_0m_e\omega^2}},$$