Pan Pearl River Delta Physics Olympiad 2019
2019 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助

Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 15 February, 2019)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。
All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。
There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。
Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。
Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。
Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。
If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。
At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Gravitational Waves (26 points) 引力波 (26 分)

Gravitational waves (GW) are the “ripples of space” predicted by Einstein in 1916. GW are transverse waves travelling at the speed of light. They are sourced by the change of mass distribution in space. In 2015, GW were detected directly by the Laser Interferometer Gravitational-Wave Observatory (LIGO). The detection is not only a verification of Einstein’s prediction after 100 years, but also provides a completely new probe of our universe and opens a new era of GW astronomy.

1916 年，爱因斯坦预言了空间的“涟漪”——引力波。引力波是横波，以光速传播，其来源是空间中物质质量分布的变化。2015 年，激光干涉引力波天文台(LIGO)发现引力波。引力波的发现不仅验证了爱因斯坦的百年预言，更是一种探测宇宙的全新手段。引力波的发现开启了引力波天文学的新时代。

In this problem, we will work in Newtonian mechanics and Newtonian gravity (instead of general relativity), and ignore the expansion of the universe, unless stated otherwise.

在本题中，除了有特殊说明之处外，我们将使用牛顿力学和牛顿引力（而不是广义相对论），且忽略宇宙膨胀。

You may find the following quantities useful here
本题中你可能用到以下数值：

Gravitational constant 牛顿引力常数	$G_N = 6.67\times10^{-11}\mathrm{m^3kg^{-1}s^{-2}}$
Speed of light 光速	$c = 3.00\times10^{8}\mathrm{m/s}$
Mass of the sun 太阳质量	$M_\odot = 1.99\times10^{30}\mathrm{kg}$
Mass of the earth 地球质量	$M_E = 5.97\times10^{24}\mathrm{kg}$
Radius of the earth 地球半径	$r_E = 6.37\times10^{6}\mathrm{m}$

Part A. Indirect Evidence of GW (10 points) 引力波的间接证据（9 分）

Before the direct discovery of GW, indirect evidence of GW has been found in a binary pulsar system in the 1970s. The binary consists of two pulsars rotating around each other in circular orbit with radius R. Let us assume each pulsar has mass M and radius r.

早在 20 世纪 70 年代，天文学家已经在双脉冲星系统中发现了引力波存在的间接证据。双脉冲星系统中，两颗脉冲星在半径为 R 的圆轨道上互相绕转。每颗脉冲星的质量为 M，半径为 r。

A1	Calculate the period of the binary system T_B. 求双星系统的运动周期 T_B。	**2 points** **2 分**

$\frac{Mv^2}{R} = \frac{G_N M^2}{4R^2}$. Thus $v = \sqrt{\frac{G_N M}{4R}}$ (1').

$T_B = \frac{2\pi R}{v} = 4\pi R \sqrt{\frac{R}{G_N M}}$ (1').

A2	Calculate the frequency of emitted gravitational wave f_{GW}. 求双星系统发射引力波的频率 f_{GW}。	**1 point** **1 分**

GW are sourced by mass distribution. When the two pulsars interchange their position (i.e. after half a period), the mass distribution returns to the same state. Thus, the period of GW is $T_B/2$ and the frequency is $f_{GW} = 2/T_B$. ($1/T_B$ for 0.5'; factor of 2 for 0.5')

A3	The binary system emits GW with power $P = \frac{2}{5c^5} \times G_N^\alpha M^\beta R^\gamma$, where c is the speed of light. Calculate the values of α, β, γ. 双星系统辐射引力波的功率为 $P = \frac{2}{5c^5} \times G_N^\alpha M^\beta R^\gamma$，其中 c 是光速。求 α，β 和 γ 的值。	**2 points** **2 分**

Dimensional analysis.
[P] = $\mathrm{m^2\ kg\ s^{-3}}$, [c^{-5}] = $\mathrm{m^{-5} s^5}$, [G_N] = $\mathrm{m^3 kg^{-1} s^{-2}}$, [M] = kg, [R] = m.
Thus,
$2 = -5 + 3\alpha + \gamma$
$1 = -\alpha + \beta$
$-3 = 5 - 2\alpha$ (1')
Thus,
$\alpha = 4$, $\beta = 5$, $\gamma = -5$ (1')

A4	After time T_C, the two pulsars collide due to GW emission. Calculate T_C. 因为引力波辐射，在 T_C 时间后，两颗脉冲星碰撞。求 T_C。	**3 points** **3 分**

The kinetic energy of the binary is $E_k = 2 \times \frac{1}{2} M v^2 = \frac{G_N M^2}{4R}$. (0.5')

The potential energy of the binary is $E_p = -\frac{G_N M^2}{2R}$. (0.5')

The total energy of the binary is $E = E_k + E_p = -\frac{G_N M^2}{4R}$. (0.5')

Thus, the change of energy over time is $\frac{dE}{dt} = \frac{G_N M^2}{4R^2}\frac{dR}{dt} = -P = -\frac{2}{5c^5} G_N^\alpha M^\beta R^\gamma = -\frac{2}{5c^5} G_N^4 M^5 R^{-5}$. (0.5') (Minus sign in the last two terms because emission of GW means energy

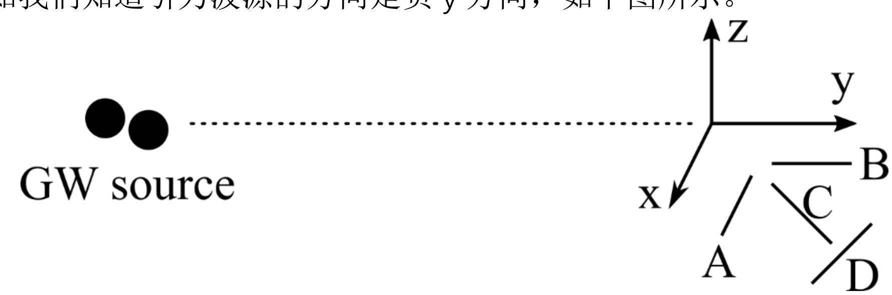
loss of the binary. We give result with both symbols α, β, γ, and their explicit values obtained in A3, to avoid double penalty. If the student is wrong in A3, he/she may still get full marks in A4.)
Integrate the above equation from R to r (corresponding time duration T_C), we get
$T_C = \frac{5c^5}{8(-\gamma-1)} G_N^{1-\alpha} M^{2-\beta}(R^{-\gamma-1} - r^{-\gamma-1}) = \frac{5c^5}{32} G_N^{-3} M^{-3}(R^4 - r^4)$. (0.5' for the integration result, and 0.5' for the correct upper/lower limits, i.e. $R^4 - r^4$ instead of R^4 only or $r^4 - R^4$)

A5	Apart from GW emission, in general relativity, there is also a correction to the gravitational potential energy $\Delta V_{GR} \propto -1/R^2$ (note the negative sign) for the pulsar system. For the same values of M, R, will this correction ΔV_{GR} increase or decrease T_B (no quantitative analysis required)? 除引力辐射外，广义相对论中，双星系统的引力势能有一个额外的修正项：$\Delta V_{GR} \propto -1/R^2$ (注意负号)。对于同样的 M 和 R，ΔV_{GR} 会增加还是减少双星系统的周期 T_B？(定性分析即可，不需要定量计算。)	**1 point 1 分**

Decrease. (1')
For fixed M, R, more negative potential → need larger velocity to provide centrifugal force → smaller period.

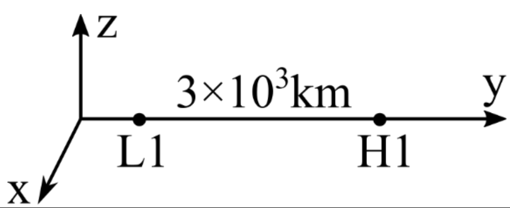
Part B. Direct Detection of GW (7 points) 引力波的发现 (7 分)

In 2016, GW were detected directly from distant merging black holes by the LIGO experiment.
2016 年，LIGO 实验组在遥远黑洞的并合事件中直接发现了引力波。

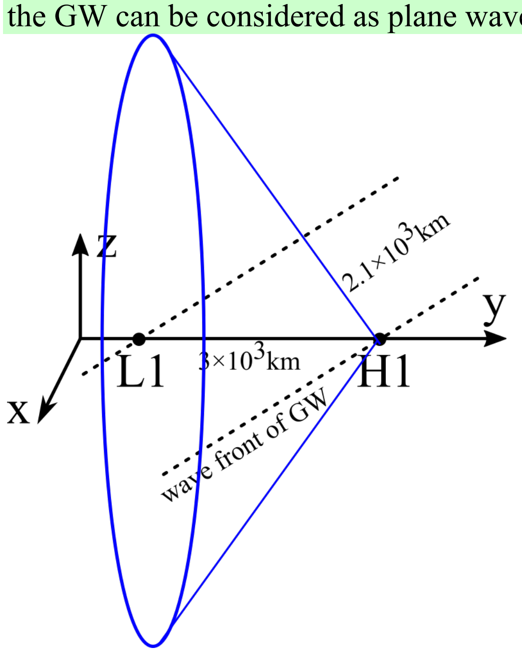
B1	Assume that we know the GW source is in the minus y direction, as shown in the below figure. 假如我们知道引力波源的方向是负 y 方向，如下图所示。  Now we measure the GW signal by measuring the change of distance of two free particles. Which of the following orientation of the particles can detect the biggest signal? Choose one from A-D below. A. Along the x-axis; B. Along the y-axis; C. Along 45 degree in the x-y plane; D. Along -45 degree in the x-y plane 我们通过测量两个自由粒子之间距离的变化来探测引力波。当粒子沿以	**1 point 1 分**

	下哪种方向放置时，测得的信号最大？从 A-D 中选择一个。 A. 沿 x 轴； B. 沿 y 轴； C. 沿 x-y 平面中 45 度角方向； D. 沿 x-y 平面中-45 度角方向	

A (Transverse wave → oscillation of spatial length is along x and z directions.) (1’)

B2	The LIGO experiment has two GW detectors L1 and H1. L1 and H1 are separated by 3×10^3km. The GW signal first arrived at L1, and arrived at H1 after 0.007s. Calculate the angle between the GW source and the L1-H1 line. Show the possible direction of the GW source in 3-dimensional space. (Draw the direction on the figure in the answer sheet.) LIGO 实验有两个探测器 L1 和 H1。L1 和 H1 相距 3×10^3km。引力波信号首先到达 L1，而后到达 H1。两者时间差为 0.007s。求引力波源与 L1-H1 连线的夹角，并画出引力波源在三维空间中的可能方位 (在答题纸上的图上画出方位)。  	**2 points** **2 分**

The distance between GW source and earth is much greater than the distance of detectors. Thus, the GW can be considered as plane waves, and a wave front can be approximated as planar.



$0.007\text{s}\times c = 2.1\times10^3\text{km}$ (0.5’)

The angle between the L1-H1 line and the GW source is thus

$\arccos 2.1/3 = 0.795$ rad $= 45.6°$ (any of these is fine) (0.5’)

Since the situation is spherically symmetric, the possible sources form a cone. Note that there is essentially no difference whether to draw the cone starting from L1, H1, or the middle point since the source is far away.

(0.5’ for a line and another 0.5’ for the whole cone)

Note: If exact spherical waves are used, and the correct result is obtained, that’s also fine. But the computation is more complicated.

B3	For the 2016 LIGO event, the binary black hole system is about 10^{22}km away from the earth. The two initial black holes are $36M_{\odot}$ and $29M_{\odot}$. The final black hole is $62M_{\odot}$. The missing mass has all emitted away as GW energy. At 1000 km away from the center of the black hole merger event, the amplitude of GW reaches $A_0 \simeq 0.01$. The energy density of GW is proportional to its amplitude squared. What's the GW energy E_R that passes through the earth? 在 2016 年 LIGO 发现的引力波事件中，双黑洞系统距地球约10^{22}km。两个黑洞的初始质量为 $36M_{\odot}$ 和 $29M_{\odot}$，合并后的黑洞质量为 $62M_{\odot}$。系统损失的质量全部以引力波能量的形式辐射出去。在距离黑洞合并事件中心 1000 km 处，引力波的振幅为 $A_0 \simeq 0.01$。引力波能量密度正比于其振幅平方。求穿过地球的引力波能量 E_R。	**2 points** **2 分**

The energy emitted: $E = \Delta M\, c^2 = (36 + 29 - 62)\times 1.99\times 10^{30}\times (3\times 10^8)^2\ \text{J} = 5.37\times 10^{47}\ \text{J}$ (1')

The energy fraction that comes to the earth: $\frac{\pi(6370)^2}{4\pi\times(10^{22})^2} = 1.01\times 10^{-37}$. (0.5')

Thus, the energy passes by the earth is 5.45×10^{10} J. (0.5')

B4	For the same conditions as given in B3, what's the amplitude A of the GW when they pass by the earth? 在与 B3 相同的条件下，求引力波到达地球时的振幅 A。	**1 point** **1 分**

Consider a shell of GW. From energy conservation, energy density of GW ~ 1/(distance)^2. (1')
Thus, the amplitude of the GW decreases inversely proportional to distance:
$A = 0.01\times \frac{1000}{10^{22}} = 10^{-21}$. (1')

B5	GW can be used to study astronomy and cosmology. Consider the expansion of the universe as an example. In the Newtonian gravity concepts, the expansion of the universe can be considered as objects running away (recession) from us. GW astronomy can be a way of studying the relation between distance and the receding velocity, and thus studying the expansion history of the universe. 我们可以用引力波来研究天文学和宇宙学。例如，研究宇宙膨胀。在牛顿引力中，宇宙膨胀可以看成是天体远离我们而去 (退行)。通过引力波天文学，可以研究天体距离与退行速度之间的关系，进而研究宇宙的膨胀历史。 For GW events with optical counterpart (for example, neutron star mergers), the receding velocity is measured by the Doppler effect. The Doppler effect of which of the following can be used to directly measure the receding velocity? Choose one from A-D below: A. Synchrotron radiation from the neutron star	**1 point** **1 分**

	B. Emission or absorption spectrum of elements C. Charged particles emitted from the neutron star system D. GW emitted from the neutron star system 对于存在光学对应体的引力波事件(例如双中子星并合)，可以通过多普勒效应测量天体的退行速度。以下哪个物理过程的多普勒效应可以直接用来测量退行速度？从 A-D 中选择一个。 A. 来自中子星的同步辐射 B. 元素的发射光谱或吸收光谱 C. 中子星系统辐射出的带电粒子 D. 中子星系统辐射出的引力波	

B. (1’) Because only the element spectrum has known frequency at emission. By comparing the observed frequency and the known emission frequency, we know the receding velocity.

Part C. Interaction between GW and Matter (10 points) 引力波和物质的相互作用 (10 分)

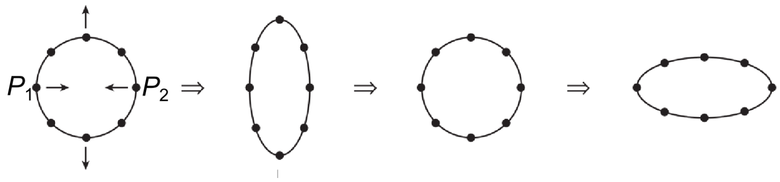
The “ripples of space” is too rough for understanding the effect of GW on matter. More concretely, one can use Newtonian physics to understand GW when its amplitude A is small (the calculation can be reproduced in general relativity in a local Lorentz frame). In Newtonian physics, the spatial length is not fluctuating. Rather, the effect of GW on matter can be considered as a periodic force proportional to $\sin(\omega_{GW}t)$ acting on matter when GW (assuming GW is plane wave with constant amplitude) passes by.

要进一步理解引力波物理，“时空的涟漪”这种说法过于粗略。更具体地，当引力波的振幅 A 很小时，我们可以在牛顿力学的框架下计算引力波的效应 (在广义相对论中，通过取局域洛伦兹系，可以验证牛顿力学的计算)。在牛顿物理中，空间距离不会有涨落变化。引力波与物质的作用体现为，引力波给物质一个正比于 $\sin(\omega_{GW}t)$ 的作用力。这里假设引力波为平面波，振幅为常数。

The amplitude A (assuming $A \ll 1$) of GW has the following effect on matter: if two free test mass particles (each has mass m) are separated by r without GW. With GW passing by its perpendicular direction (throughout Part C, we assume the propagation direction of GW is perpendicular to the line of the two particles), their distance changes from $r(1-A)$ to $r(1+A)$ periodically. The oscillation of test particle is of the pattern below (we draw many test particles to show the effect of GW more clearly, but in the problem let us just consider the two particles P_1 and P_2).

振幅为 A (假设 $A \ll 1$) 的引力波会为物质带来如下效应：考虑两个自由的检验粒子，每个粒子的质量都为 m。当没有引力波通过时，两个粒子之间的距离为 r。当有引力波沿垂直于粒子连线的方向通过时 (整个 C 部分中，均考虑引力波传播方向与两粒子连线方向垂直)，两个粒子之间的距离在 $r(1-A)$ 与 $r(1+A)$ 之间周期性变化。检验粒子的振荡如下图所

示。（下图中画了多个检验粒子的运动，以更清楚地显示引力波的作用。在本题中，我们只考虑两个粒子P_1和P_2。）



C1	Now, connect the two particles with a spring with spring constant k and unstretched length r (i.e. the spring does not change the initial distance between particles without GW). Assume that the spring is light and the force that GW acts on the spring is negligible. For $k \gg m\,\omega_{GW}^2$, calculate the oscillation amplitude A' between these two particles when GW with amplitude A pass by, such that the distance between the two particles change between $r(1-A')$ and $r(1+A')$. Note: Here we assume the two particles have minimal kinetic energy as allowed in the above setup. (Otherwise additional kinetic energy can cause oscillations with larger amplitudes for the spring.) 现在，将两个粒子用弹簧连接起来。弹簧的弹性系数为 k，原长为 r（也就是说，没有引力波通过时，弹簧不改变粒子的初始距离）。假设弹簧很轻，引力波作用在弹簧上的力可以忽略。在 $k \gg m\,\omega_{GW}^2$ 的情况下，计算振幅为 A 的引力波通过时，粒子的振幅 A'。这里 A'的定义为，引力波通过时，两粒子距离在$r(1-A')$ 和 $r(1+A')$ 之间变化。 注：假设两粒子的动能是可以满足题设条件的最小动能。（否则，额外的动能可以导致额外的振动，以及更大的振幅。）	**3 points** **3 分**

For free test particles separated by r, let the position of each particle is $x = \pm r/2$, and let's study the particle with $x = r/2$. Let the periodic force be $F = F_0 \sin(\omega_{GW} t)$ acting on this particle.
Integrate $F = m\ddot{x}$ (0.5') twice (or you first guess the form of x, and determine coefficients by taking derivative), we get $x = \frac{r}{2} - \frac{F_0 \sin(\omega_{GW} t)}{m\,\omega_{GW}^2}$. (0.5')
Compare it with the definition of A, we get $F_0 = \frac{Ar}{2} m\omega_{GW}^2$. (0.5')
An opposite force will act on the particle located at $x = -r/2$.
Now, for the two particles connected by a spring:
The condition $k \gg m\omega_{GW}^2$ indicates that the frequency of GW is very slow. Slowly acting a force on the spring, the length of the spring just follows the force (0.5'). The maximal length that the spring stretches is $r + \frac{F_0}{k} = r + \frac{Ar}{2k} m\omega_{GW}^2$ (0.5'), i.e. $A' = \frac{A}{2k} m\omega_{GW}^2$ (0.5')

=============

Alternative solution of C1 - C2: Instead of considering half spring, consider the full spring:

Let $x_1 = \frac{r}{2} + x_{10} \sin \omega_{GW} t$ and $x_2 = -\frac{r}{2} + x_{20} \sin \omega_{GW} t$.

$$\begin{pmatrix} -m\omega_{GW}^2 & 0 \\ 0 & -m\omega_{GW}^2 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} + \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$
$$\begin{pmatrix} k - m\omega_{GW}^2 & -k \\ -k & k - m\omega_{GW}^2 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$
$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} k - m\omega_{GW}^2 & -k \\ -k & k - m\omega_{GW}^2 \end{pmatrix}^{-1} \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$
$$= \frac{1}{(k - m\omega_{GW}^2)^2 - k^2} \begin{pmatrix} k - m\omega_{GW}^2 & k \\ k & k - m\omega_{GW}^2 \end{pmatrix} \begin{pmatrix} F_0 \\ -F_0 \end{pmatrix}$$

Distance between the two particles:

$$d = x_1 - x_2 = r + (x_{10} - x_{20}) \sin \omega_{GW} t = r - \frac{2m\omega_{GW}^2 F_0}{(k - m\omega^2)^2 - k^2} \sin \omega_{GW} t$$
$$d = r \left[1 - \frac{(m\omega_{GW}^2)^2 A}{(k - m\omega_{GW}^2)^2 - k^2} \right] \sin \omega t$$

C1: When $k \gg m\omega^2$, $A' \approx \frac{(m\omega_{GW}^2)^2 A}{2km\omega_{GW}} = \frac{m\omega_{GW}^2}{2k} A$.

C2: When $k \ll m\omega^2$, $A' \approx \frac{(m\omega_{GW}^2)^2 A}{(m\omega_{GW}^2)^2 - 2km\omega_{GW}^2} \approx A + \frac{2k}{m\omega_{GW}^2} A$.

C2	For the same setup as Part C1, but $k \ll m\,\omega_{GW}^2$, calculate A'. Keep the linear terms in k, and the higher order terms in k can be neglected (i.e. if the exact result is $A + Bk + Ck^2 + \cdots$, we require that you get $A + Bk$ and you can neglect higher terms Ck^2 and so on). 设 $k \ll m\,\omega_{GW}^2$，其他条件与 C1 中相同，计算 A'。这里，只需保留至 k 的线性阶，k 的高阶项可以忽略。（也就是说，如果精确结果是 $A + Bk + Ck^2 + \cdots$，则你需要写出 $A + Bk$。你可以忽略 Ck^2 等高阶项。）	**3 points** **3 分**

From symmetry, the center of the spring will not move. Thus, let's consider half spring from the center to the particle to the right. The spring constant of half spring is $2k$ (0.5').

The Newtonian 2nd law: $m\ddot{x} = F_0 \sin(\omega_{GW} t) - 2k \left(x - \frac{r}{2} \right)$. (*) (0.5')

In principle, this equation (*) can be solved exactly without assuming the condition between k and $m\omega_{GW}^2$. But as we do not require to solve complicated differential equations, here we just work out the solution assuming small k.

Let $x = x^{(0)} + x^{(1)} + x^{(2)} + \cdots$. Where the superscript indicates the order in k. (0.5') We will ignore $x^{(2)}$ and higher terms.

We have already solved $x^{(0)} = \frac{r}{2} - \frac{Ar\,sin(\omega_{GW}t)}{2}$ (free particle with $k = 0$). Insert it into (*), we get

$\ddot{x}^{(1)} = \frac{kAr\sin(\omega_{GW}t)}{m}$. (0.5')

Similar to the case in C1, integrate this equation twice, we get

$x^{(1)} = -\frac{kAr\sin(\omega_{GW}t)}{m\omega_{GW}^2}$. (0.5')

Thus, up to linear order in k, the maximal distance is the maximal of $2(x^{(0)} + x^{(1)})$. The factor of two is because we have two particles at $x = \pm r/2$. Inserting the above results, we have

$A' = A + \frac{2kA}{m\omega_{GW}^2}$. (0.5')

Note: If you directly get the exact solution of (*), it is also correct. The form is

$A' = \frac{A}{1-\frac{2k}{m\omega_{GW}^2}}$.

C3	For the same setup as Part C1, but $k = m\,\omega_{GW}^2/2$, qualitatively describe how the distance between two particles changes with time. No explicit calculation is needed. 设 $k = m\,\omega_{GW}^2/2$，其他条件与 C1 中相同，定性描述两粒子间的距离将如何随时间变化。无需定量计算。	**1 point** **1 分**

Resonance happens. The distance oscillates (0.5') and the amplitude of oscillation grows with time (0.5').

C4	We'd like to estimate when GW pass through the earth, how much GW energy the earth can absorb. The earth is a system that the pressure of matter balances self-gravity. The real earth is too complicated but let's consider a toy model of the earth, as two particles at rest separated by $r = 6000$ km, each particle has mass $m = 3\times10^{24}$ kg. The self-gravity between these two particles are balanced by force provided by a light spring connecting these particles. And the unstretched length of the spring is 7000 km if no force acts on it. For the GW signal described by Part B3 and B4, with frequency $f = 100$Hz, estimate the order-of-magnitude of energy absorbed by the earth from one period of GW oscillation. 我们将估计当引力波穿过地球时，地球可以吸收多少能量。地球是一个自身压强与引力平衡的系统。真实的地球非常复杂。这里我们考虑一个地球的玩具模型：考虑两个静止粒子相距 $r = 6000$ km。每个粒子的质量为 $m = 3\times10^{24}$ kg。两个粒子之间由一根轻弹簧连接。弹簧的弹力与两粒子之间的引力平衡。假如不受力，弹簧的原长为7000 km。对于 B3、B4 中描述的，频率为 $f = 100$ Hz的引力波信号，求在引力波的一个振荡周期中，地球吸收的能量 (估计数量级即可)。	**2 points** **2 分**

First estimate the spring constant of the earth: $F_{\text{grav}} = k\Delta r = \frac{G_N m^2}{r^2}$. Thus, $k = 1.67\times10^{19}\text{kg/s}^2$. (0.5')
Now determine the spring constant is for which case (C1-C3):
$m\omega_{GW}^2 = 3\times10^{24}\text{kg}\times(100\times2\pi\text{ Hz})^2 \gg k$. Thus, we can use the case of C2. (0.5')
For order-of-magnitude estimate, the size of the earth simply change by an amount $\delta r = rA = 6\times10^{-15}\text{m}$.
For each period, GW first slowly stretch the earth to do work, and then the work is dissipated to the earth (for example, the heat in the spring in this case) when the spring returns.
The force is about $F_0 = \frac{1}{2} m\omega_{GW}^2 \text{A}r = 3.5\times10^{15} N$ (0.5')
The work GW does in each period is $W = F_0\delta r \sim 21$ J. (0.5') (To be accurate, there is an average of $\sin(\omega_{GW})\cos(\omega_{GW})$ over two of 1/4 periods. But here we are only interested in the order-of-magnitude.)

C5	Black holes are so dense objects that even objects travelling at the speed of light (such as GW) cannot escape. For a black hole with mass the same as that of the earth (and at the same location as the earth), calculate the amount of GW energy that the black hole absorbs for the event described in Part B3 and B4. 黑洞是一种极端致密的天体：即使以光速运动的物体 (如引力波) 也不能逃出黑洞。考虑一个黑洞，与地球有同样的质量，并处于与地球同样的位置。对于 B3、B4 中描述的引力波信号，求此黑洞吸收的引力波能量。	**1 point 1 分**

Escape velocity: $v = \sqrt{\frac{2G_N m}{r}} = c$. (0.5') Thus, $r = \frac{2G_N m}{c^2} = 0.0088\text{m}$. (0.5')
Note: using Newtonian mechanics instead of relativity, we are actually making two mistakes: (1) light & GW are actually massless particles instead of massive ones, and their momentum-energy relations are relativistic; and (2) the Newtonian gravitational potential is not enough to describe gravity. Coincidentally, these two mistakes cancel each other's effect, and $r = \frac{2G_N m}{c^2}$ actually holds even in general relativity. But in general relativity, even if GW did not touch the horizon $r = \frac{2G_N m}{c^2}$, but instead reaches $r = \frac{3G_N m}{c^2}$ (photon sphere), GW will eventually fall into the black hole. Thus $r = \frac{3G_N m}{c^2}$ is also considered correct, though it does not follow from Newtonian mechanics.
Compared to problem B3, the GW energy come to the black hole is $5\times10^{10}\text{ J}\times\frac{0.0088^2}{(6\times10^6)^2} = 1.04\times10^{-7}$ J. (0.5') Since GW cannot escape the black hole, this is the GW energy that the black hole absorbs. (0.5')

If you use $E = h\nu$ for the graviton energy, or use angular momentum conservation to calculate the condition for the GW to be absorbed, they can also be considered correct. There is an unique answer in general relativity, but you can use different ways to model it in Newtonian mechanics.

END of Problem 1
问题 1 完

Problem 2 Synchronization (34 marks) 同步（34 分）

Synchronization is a very common physical phenomenon. As early as in the 17th century, the famous Dutch scientist Christiaan Huygens observed that when two pendulum clocks are suspended from a common beam, they tend to oscillate in synchrony. In part A of this problem, we will consider a model of this phenomenon. In part B of this problem, we will consider a modern example of synchronization. Students can work on either part first before working on the other part.
同步是一种非常常见的物理现象。早在 17 世纪，著名的荷兰科学家克里斯蒂安•惠更斯就观察到，当两个单摆时钟悬挂在同一根梁上时，它们往往会同步振荡。在这个问题的 A 部份中，我们将考虑这种现象的一个模型。在这个问题的 B 部份中，我们将考虑一个现代的同步示例。同学可先完成任一部分，再完成另一部份。

A. The Pendulums (23 marks) 單擺 (23 分)

A single pendulum consists of a bob with mass m suspended vertically from a fixed point with a massless string of length L, subject to gravitational acceleration g. Let $q(t)$ be the angular displacement of the pendulum from the vertical at time t. When the bob moves, it encounters a constant frictional force of magnitude mLb in the opposite direction of motion.
单摆由一个质量m的小物块组成，物块从一固定点用绳子垂直悬挂，并受到重力加速度g的影响。绳子没有质量，长度为L。 在时间t，设单摆从垂直方向的角位移为$q(t)$。当物块移动时，有一数值为 mLb 的恒定摩擦力作用于它运动的反方向上。

A1	Write the dynamical equation of $q(t)$ for small oscillations. 写出小振动時$q(t)$的动力学方程式。	**2 points** **2 分**

Remark: To keep your equation simple, you may introduce the angular frequency
备注：为了使方程式更简洁，您可引入角频率

$$\omega = \sqrt{\frac{g}{L}},$$

and use the sign function defined by
并使用下面所定义的正負函数

$$\mathrm{sign} f = \begin{cases} 1 \text{ for } f > 0, \\ 0 \text{ for } f = 0, \\ -1 \text{ for } f < 0. \end{cases}$$

Using Newton's law of motion,

$$mL\frac{d^2q}{dt^2} = -mLb\,\mathrm{sign}\left(\frac{dq}{dt}\right) - mg\sin\,q,$$

For small oscillations, $\sin q \approx q$. Hence

$$\ddot{q} = -b\ \mathrm{sign}\dot{q} - \omega^2 q,$$

where $\ddot{q} = d^2q/dt^2$ and $\dot{q} = dq/dt$.

To compensate the loss of kinetic energy due to the friction in each cycle, the pendulum receives a kick every cycle. To simplify the calculations, we assume that the kick takes place when $q = -b/\omega^2$ and its angular velocity is positive.

为了补偿于每个周期中由摩擦力所引致的动能损失，单摆每个周期都会受到一次踢动。为了简化计算，我们假设踢动发生在$q = -b/\omega^2$并且当它的角速度为正时。

A2	Suppose that the angular velocity of the pendulum is u_n immediately after the n^{th} kick. Calculate $q(t)$ and $\dot{q}(t)$ in the cycle after the n^{th} kick. For convenience, we choose $t = 0$ at the n^{th} kick in this part and below. 假设单摆的角速度在第n次踢动后的瞬间为u_n。在第n次踢动后的周期中，计算$q(t)$和$\dot{q}(t)$。为方便起见，在此部份及以下部份中，我们设在第n次踢动的时间为$t = 0$。 For clarity, give your answer in three parts: 为清楚起见，请分三部份给出答案： (a) The first quarter of the cycle, (a) 第一个四分之一的周期， (b) the second and third quarters of the cycle, (b) 第二和第三个四分之一的周期， (c) the fourth quarter of the cycle. (c) 第四个四分之一的周期。	**2+2+2 points 2+2+2 分**

(A2a) In the first quarter of the cycle after the n^{th} kick, the angular velocity is positive. Hence

$$\ddot{q} = -\omega^2\left(q + \frac{b}{\omega^2}\right).$$

The solution is a simple harmonic motion centered at $q = -b/\omega^2$. Hence the solution takes the form

$$q(t) = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t,$$

and

$$\dot{q}(t) = u_n\cos\omega t.$$

(A2b) In the second and third quarters of the cycle, the angular velocity is negative. Hence

$$\ddot{q} = -\omega^2\left(q - \frac{b}{\omega^2}\right).$$

The solution is a simple harmonic motion centered at $q = b/\omega^2$. The initial condition at $t = \pi/2\omega$ is $q\left(\frac{\pi}{2\omega}\right) = -\frac{b}{\omega^2} + \frac{u_n}{\omega^2}$ and $\dot{q}\left(\frac{\pi}{2\omega}\right) = 0$. Hence the solution takes the form

$$q(t) = \frac{b}{\omega^2} + \left[q\left(\frac{\pi}{2\omega}\right) - \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{\pi}{2}\right),$$
$$q(t) = \frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{2b}{\omega^2}\right)\sin\omega t,$$

and

$$\dot{q}(t) = \left(u_n - \frac{2b}{\omega}\right)\cos\omega t.$$

(A2c) In the fourth quarter of the cycle, the angular velocity is positive. Hence

$$\ddot{q} = -\omega^2\left(q + \frac{b}{\omega^2}\right).$$

The solution is a simple harmonic motion centered at $q = -b/\omega^2$. The initial condition at $t = 3\pi/2\omega$ is $q\left(\frac{3\pi}{2\omega}\right) = -\frac{u_n}{\omega} + \frac{3b}{\omega^2}$ and $\dot{q}\left(\frac{\pi}{2\omega}\right) = 0$. Hence the solution takes the form

$$q(t) = -\frac{b}{\omega^2} + \left[q\left(\frac{3\pi}{2\omega}\right) + \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{3\pi}{2}\right),$$
$$q(t) = -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t,$$

and

$$\dot{q}(t) = \left(u_n - \frac{4b}{\omega}\right)\cos\omega t.$$

A3	Suppose at each kick, a fixed amount of kinetic energy of the magnitude $mL^2h^2/2$ is injected to the pendulum, where h has the dimension of an angular velocity. Calculate the relation between u_{n+1} and u_n. 假设在每次踢动时，有数值为 $mL^2h^2/2$ 的动能被注入到单摆中，其中h具有角速度的量纲。计算u_{n+1}和之u_n间的关系。	**2 points** **2 分**

At the end of the n^{th} cycle,

$$\dot{q}(t) = u_n - \frac{4b}{\omega}.$$

At the beginning of the $(n+1)^{\text{th}}$ cycle,

$$\frac{1}{2}mL^2u_{n+1}^2 = \frac{1}{2}mL^2\left(u_n - \frac{4b}{\omega}\right)^2 + \frac{1}{2}mL^2h^2.$$

Therefore

$$u_{n+1} = \sqrt{\left(u_n - \frac{4b}{\omega}\right)^2 + h^2}.$$

A4	What is the value of u_n after many kicks? 经过多次踢动后，u_n的数值是什么？	**2 points** **2 分**

After many kicks, $u_{n+1} = u_n$. Hence

$$u_n^2 = \left(u_n - \frac{4b}{\omega}\right)^2 + h^2.$$
$$u_n = \frac{h^2\omega}{8b} + \frac{2b}{\omega}.$$

A5	Suppose that at time t_0 during the first quarter of the cycle after the n^{th} kick, the pendulum receives an angular impulse equal to $mL^2\alpha$. Calculate the time at which: 假设在第n次踢动后的第一个四分之一周期内，單摆接受了数值为 $mL^2\alpha$ 的角冲量。计算以下情况的时间： (a)the friction changes sign the first time, (a) 摩擦力第一次改变方向时， (b) the friction changes sign the second time, (b) 摩擦力第二次改变方向时， (c) the pendulum receives the $(n+1)^{\text{th}}$ kick. (c) 單摆受到第$n+1$次踢動时。 Give your answer to the first order in α. 答案的表达式展开至α的第一阶。	**3 points** **3 分**

(A5a) After the pendulum has received the angular impulse at time t_0, the pendulum motion takes the form

$$q(t) = -\frac{b}{\omega^2} + \left(q(t_0) + \frac{b}{\omega^2}\right)\cos(\omega t - \omega t_0) + \frac{\dot{q}(t_0)}{\omega}\sin(\omega t - \omega t_0).$$

Since $q(t_0) = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t_0$ and $\dot{q}(t_0) = u_n\cos\omega t_0 + \alpha$, we have

$$\begin{gathered} q(t) = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t_0\cos(\omega t - \omega t_0) + \frac{u_n}{\omega}\cos\omega t_0\sin(\omega t - \omega t_0) + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0) \\ = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0). \\ \dot{q}(t) = u_n\cos\omega t + \alpha\cos(\omega t - \omega t_0). \end{gathered}$$

The friction changes sign when $\dot{q}(t) = 0$. Suppose this takes place when $\omega t = \frac{\pi}{2} + \varepsilon$. Then

$$\begin{gathered} 0 = u_n\cos\left(\frac{\pi}{2} + \varepsilon\right) + \alpha\cos\left(\frac{\pi}{2} + \varepsilon - \omega t_0\right). \\ \varepsilon \approx \frac{\alpha\sin\omega t_0}{u_n}. \\ t = \frac{\pi}{2\omega} + \frac{\alpha\sin\omega t_0}{\omega u_n}. \end{gathered}$$

(A5b) After the friction has changed sign the first time, the pendulum motion takes the form

$$q(t) = \frac{b}{\omega^2} + \left[q\left(\frac{\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) - \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{\pi}{2} - \varepsilon\right).$$

Note that

$$\begin{aligned} q\left(\frac{\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) &= -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\left(\frac{\pi}{2} + \varepsilon\right) + \frac{\alpha}{\omega}\sin\left(\frac{\pi}{2} + \varepsilon - \omega t_0\right) \\ &\approx -\frac{b}{\omega^2} + \frac{u_n}{\omega}\cos\varepsilon + \frac{\alpha}{\omega}\cos(\omega t_0 - \varepsilon) \approx \frac{u_n}{\omega} - \frac{b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0. \end{aligned}$$

Hence

$$q(t) = \frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{2b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0\right)\cos\left(\omega t - \frac{\pi}{2} - \varepsilon\right).$$

$$\dot{q}(t) = -\left(u_n - \frac{2b}{\omega} + \alpha\cos\omega t_0\right)\sin\left(\omega t - \frac{\pi}{2} - \varepsilon\right).$$

The friction changes sign the second time when $\dot{q}(t) = 0$. This takes place when $\omega t - \frac{\pi}{2} - \varepsilon = \pi$. Hence

$$t = \frac{3\pi}{2\omega} + \frac{\alpha\sin\omega t_0}{\omega u_n}.$$

(A5c) After the friction has changed sign the second time, the pendulum motion takes the form

$$q(t) = -\frac{b}{\omega^2} + \left[q\left(\frac{3\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) + \frac{b}{\omega^2}\right]\cos\left(\omega t - \frac{3\pi}{2} - \varepsilon\right).$$

Note that

$$q\left(\frac{3\pi}{2\omega} + \frac{\varepsilon}{\omega}\right) = \frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{2b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0\right)\cos\left(\frac{3\pi}{2} + \varepsilon - \frac{\pi}{2} - \varepsilon\right) = -\frac{u_n}{\omega} + \frac{3b}{\omega^2} - \frac{\alpha}{\omega}\cos\omega t_0.$$

Hence

$$q(t) = -\frac{b}{\omega^2} - \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2} + \frac{\alpha}{\omega}\cos\omega t_0\right)\cos\left(\omega t - \frac{3\pi}{2} - \varepsilon\right).$$

The pendulum receives the $(n+1)^{\text{th}}$ kick when $q(t) = -b/\omega^2$. This takes place when $\omega t - \frac{3\pi}{2} - \varepsilon = \frac{\pi}{2}$. Hence

$$t = \frac{2\pi}{\omega} + \frac{\alpha\sin\omega t_0}{\omega u_n}.$$

A6	Suppose that the time t_0 at which the pendulum receives an angular impulse equal to $mL^2\alpha$ is in the fourth quarter of the cycle after the n^{th} kick instead of the first quarter. Calculate the time at which the pendulum receives the $(n+1)^{\text{th}}$ kick. Give your answer to the first order in α. 假设单摆接受角冲量 $mL^2\alpha$ 的时间t_0是在第n次踢动后的第四个四分之一而不是第一个四分之一的周期内。计算单摆接受第$n+1$次踢動的时间。答案的表达式展开至α的第一阶。	**2 points** **2 分**

After the pendulum has received the angular impulse at time t_0, the pendulum motion takes the form

$$q(t) = -\frac{b}{\omega^2} + \left(q(t_0) + \frac{b}{\omega^2}\right)\cos(\omega t - \omega t_0) + \frac{\dot{q}(t_0)}{\omega}\sin(\omega t - \omega t_0).$$

Since $q(t_0) = -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t_0$ and $\dot{q}(t_0) = \left(u_n - \frac{4b}{\omega}\right)\cos\omega t_0 + \alpha$, we have

$$q(t) = -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t_0 \cos(\omega t - \omega t_0) + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\cos\omega t_0 \sin(\omega t - \omega t_0) + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0)$$

$$= -\frac{b}{\omega^2} + \left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin\omega t + \frac{\alpha}{\omega}\sin(\omega t - \omega t_0).$$

$$\dot{q}(t) = \left(u_n - \frac{4b}{\omega}\right)\cos\omega t + \alpha\cos(\omega t - \omega t_0).$$

The pendulum receives the $(n+1)^{\text{th}}$ kick when $q(t) = -b/\omega^2$. Suppose this takes place when $\omega t = 2\pi + \delta$. Then

$$\left(\frac{u_n}{\omega} - \frac{4b}{\omega^2}\right)\sin(2\pi + \delta) + \frac{\alpha}{\omega}\sin(2\pi + \delta - \omega t_0) = 0.$$

$$\delta \approx \frac{\alpha \sin\omega t_0}{u_n - \frac{4b}{\omega}}.$$

$$t = \frac{2\pi}{\omega} + \frac{\alpha\sin\omega t_0}{\omega\left(u_n - \frac{4b}{\omega}\right)}.$$

Now consider two pendulum clocks. Let $q_1(t)$ and $q_2(t)$ be the angular displacements of the two clocks. The bob mass m, length L, friction parameter b and kick size h of the two pendulums are identical. Suppose that when $q_2 = -b/\omega^2$, pendulum 2 sends a small angular impulse equal to $mL^2\alpha$ on pendulum 1, and when $q_1 = -b/\omega^2$, pendulum 1 sends a small angular impulse equal to $mL^2\alpha$ on pendulum 2. (Here, $\alpha > 0$.)
现在考虑两个单摆。设两个单摆的角位移分别為$q_1(t)$和$q_2(t)$。两个单摆的小物塊质量m、长度L、摩擦参数b和踢動的大小h均是相同。假设当$q_2 = -b/\omega^2$时，单摆 2 发出一个数值為$mL^2\alpha$的小角冲量給单摆 1。当$q_1 = -b/\omega^2$时，单摆 1 发出一个数值為$mL^2\alpha$的小角冲量給单摆 2 。（这里 $\alpha > 0$。）

A7	Suppose the phase lag of pendulum 2 relative to pendulum 1 is ϕ_n at the beginning of n^{th} cycle of pendulum 1, and $0 \le \phi_n < \pi/2$. Calculate the relation between ϕ_{n+1} and ϕ_n. 假设在单摆 1 第n个周期的开始时，单摆 2 相对于单摆 1 的相位滞后為ϕ_n，其中$0 \le \phi_n < \pi/2$。 试计算ϕ_{n+1}和ϕ_n之间的关系。	**4 points** **4 分**

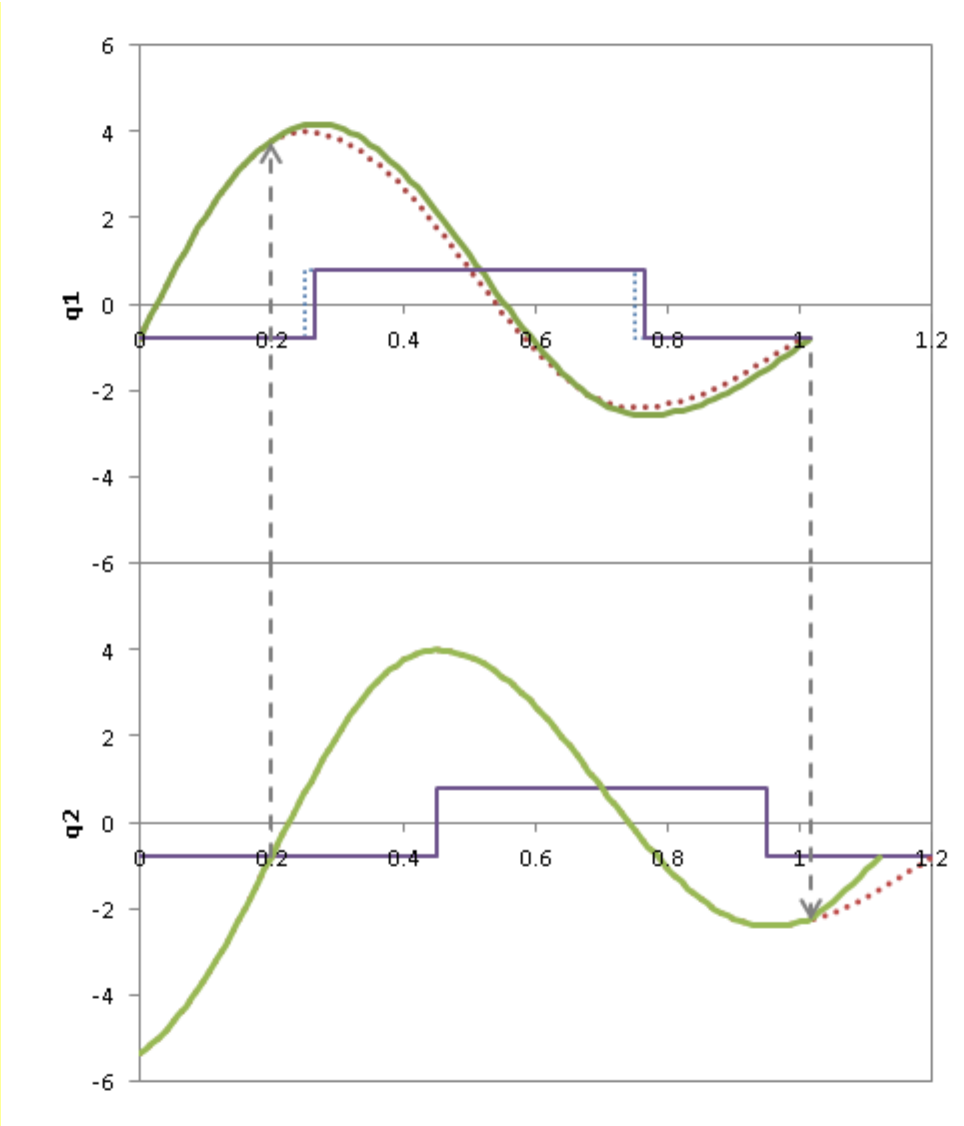


Figure: An example of how the phase lag of pendulum 2 relative to pendulum 1 is reduced in a cycle. Initially, the phase lag is 0.2 cycle. At $\omega t = 0.2$, pendulum 2 sends an angular impulse to pendulum 1. The angular velocity of pendulum 1 increases, causing the instant of the next kick to postpone from $\omega t = 1$ to $\omega t = 1.02$. At $\omega t = 1.02$, pendulum 1 sends an angular impulse to pendulum 2. The angular velocity of pendulum 2 increases, causing the instant of the next kick to move forward from $\omega t = 1.2$ to $\omega t = 1.12$. Hence the phase lag is reduced to $1.12 - 1.02 = 0.1$ cycle.

$$q_1 = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin\omega t,$$
$$q_2 = -\frac{b}{\omega^2} + \frac{u_n}{\omega}\sin(\omega t - \phi_n).$$

At time $t_0 = \phi_n/\omega$, pendulum 2 sends an angular impulse on pendulum 1. At this instant, the phase of pendulum 1 is in the first quarter of the cycle. Using the result of (A5), pendulum 1 receives the next kick at

$$t_1 = \frac{2\pi}{\omega} + \frac{\alpha \sin \phi_n}{\omega u_n}.$$

At this instant, pendulum 1 sends an angular impulse on pendulum 2, and the phase of pendulum 2 is in the fourth quarter of the cycle. Using the result of (A6), pendulum 2 receives the next kick at

$$t_2 - \frac{\phi_n}{\omega} = \frac{2\pi}{\omega} + \frac{\alpha \sin(\omega t_1 - \phi_n)}{\omega \left(u_n - \frac{4b}{\omega}\right)} \approx \frac{2\pi}{\omega} - \frac{\alpha \sin \phi_n}{\omega \left(u_n - \frac{4b}{\omega}\right)}.$$

Phase difference:

$$\omega t_2 - \omega t_1 = 2\pi + \phi_n - \frac{\alpha \sin \phi_n}{u_n - \frac{4b}{\omega}} - 2\pi - \frac{\alpha \sin \phi_n}{u_n}.$$

$$\phi_{n+1} = \phi_n - \frac{\alpha \sin \phi_n}{u_n - \frac{4b}{\omega}} - \frac{\alpha \sin \phi_n}{u_n}.$$

A8	When ϕ_n is very small, calculate the number of cycles for ϕ_n to reduce by a factor of 10. 当ϕ_n非常少時，试计算ϕ_n减少 10 倍所需的周期数。	**2 points** **2 分**

When ϕ_n is very small,

$$\phi_n = \phi_{n-1} - \left(u_n - \frac{4b}{\omega}\right)^{-1} \alpha \sin \phi_{n-1} - u_n^{-1} \alpha \sin \phi_{n-1} \approx \left[1 - \alpha \left(u_n - \frac{4b}{\omega}\right)^{-1} - \alpha u_n^{-1}\right] \phi_{n-1}.$$

Note that for the system to sustain many kicks, $u_n - \frac{4b}{\omega} > 0$. For ϕ_n to reduce by a factor of 10,

$$\frac{1}{10} = \left[1 - \alpha \left(u_n - \frac{4b}{\omega}\right)^{-1} - \alpha u_n^{-1}\right]^N \approx \exp\left\{-\left[\left(u_n - \frac{4b}{\omega}\right)^{-1} + u_n^{-1}\right] \alpha N\right\},$$

$$N \approx \frac{1}{\alpha} \left[\left(u_n - \frac{4b}{\omega}\right)^{-1} + u_n^{-1}\right]^{-1} \ln 10.$$

Remarks: This part of the problem is adopted from [1]. In that reference, the angular impulses are negative, leading to the two clocks synchronizing oppositely. This agrees with Huygens' observation. On the other hand, there are experiments such as metronomes placed on flexible platforms that show congruent synchronization.

[1] H. M. Oliveira and L. V. Melo, Huygens synchronization of two clocks, Sci. Rep. 5: 11548 (2015).

B. The Power Grid (11 marks) 电网 (11 分)

Synchronization is an important concept in the transmission of electricity in the power grid. The power grid is a network of nodes and links. Each node is an electric generator or other power consumption devices. The links are the transmission cables. Electric power is transmitted in the alternating current (AC) mode at 50 Hz or 60 Hz at a fixed voltage. However, the AC voltage of each node in the network has a slightly different phase.

同步是电网输电中的一个重要概念。电网是一个由节点和链路组成的网络。每个节点是一部发电机或其他功耗设备。链路是传输电缆。电力运用交流电（AC）模式以 50 Hz 或 60 Hz 频率经固定电压传输。但是，网络中每个节点的交流电压具有略微不同的相位。

B1	Consider a transmission cable connecting nodes 1 and 2. The inductance of the cable is L. The electric potentials of nodes 1 and 2 are $V_j(t) = V\cos(\omega t + \theta_j)$ for $j = 1, 2$. Calculate the time-averaged power transmitted from node 1 to 2. You may neglect the time dependence of θ_j. 考虑连接节点 1 和 2 的传输电缆。电缆的电感是L。节点 1 和 2 的电势为 $V_j(t) = V\cos(\omega t + \theta_j)$，其中$j = 1, 2$。计算从节点 1 传输到节点 2 的时间平均功率。你可以忽略θ_j的时间依赖性。	**3 points** **3 分**

Potential difference of node 1 relative to 2:

$$V(t) = V\cos(\omega t + \theta_1) - V\cos(\omega t + \theta_2) = -2V\sin\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right)\sin\left(\frac{\theta_1 - \theta_2}{2}\right).$$

Current from node 1 to 2: $V(t) = L\frac{dI(t)}{dt}$ implies

$$I(t) = -\int 2V\sin\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right)\sin\left(\frac{\theta_1 - \theta_2}{2}\right)dt = \frac{2V}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right).$$

Power transmitted from node 1 to 2:

$$P(t) = I(t)V\cos(\omega t + \theta_1) = \frac{2V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\omega t + \frac{\theta_1 + \theta_2}{2}\right)\cos(\omega t + \theta_1) =$$
$$= \frac{V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\left[\cos\left(\frac{\theta_1 - \theta_2}{2}\right) + \cos\left(2\omega t + \frac{3\theta_1 + \theta_2}{2}\right)\right].$$

Average power from 1 to 2:

$$\langle P\rangle = \frac{V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\left[\cos\left(\frac{\theta_1 - \theta_2}{2}\right) + \left\langle\cos\left(2\omega t + \frac{3\theta_1 + \theta_2}{2}\right)\right\rangle\right]$$
$$= \frac{V^2}{\omega L}\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{V^2}{2\omega L}\sin(\theta_1 - \theta_2).$$

A network of electric generators and motors, labeled $j = 1, 2, \ldots, N$, are connected with each other. Their electric potentials are $V_j(t) = V\cos(\omega t + \theta_j)$ for $j = 1, 2, \ldots, N$, and the inductances

of the connecting cables are L. The generator or motor at node j rotates with the phase angle $\omega t+\theta_j$ and its moment of inertia is I. The external power source or drain is P_j ($P_j>0$ if j is a generator, and $P_j<0$ if j is a motor). At the same time, the power dissipation due to friction is given by $\kappa(\omega+\dot{\theta}_j)^2/2$ at node j.
一个网络，由发电机和电动机彼此完全连接而成，发电机和电动机的标记為$j=1,2,\ldots,N$。它们的电势為$V_j(t)=V\cos(\omega t+\theta_j)$，其中$j=1,2,\ldots,N$。用于连接它们各点之间的电缆，其电感为$L$。节点处的发电机或电动机以相角$\omega t+\theta_j$旋转，其转动惯量为$I$。外部电能的供应或消耗為$P_j$（如果$j$是发电机，$P_j>0$ 。如果j是电动机，$P_j<0$）。同时，在节点j由摩擦引起的功耗是$\kappa\left(\omega+\dot{\theta}_j\right)^2/2$。

B2	Derive the dynamical equation for θ_j as a function of time. Assume that the rates of change of θ_j are much less than ω, such that the dynamical equation can be approximated by retaining only terms up to the first order of θ_j. 推导出θ_j以时间為函数的动力学方程。可假设θ_j的变化率远小于ω，使得动力学方程式可通过仅保留至θ_j的第一阶项来近似。	**2 points** **2 分**

Using the conservation of power,

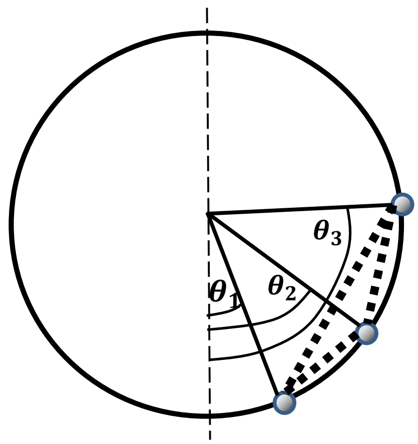
$$
\begin{aligned}
P_j &= \frac{d}{dt}\left[\frac{I}{2}\left(\omega+\dot{\theta}_j\right)^2\right]+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_j-\theta_k\right)+\frac{\kappa}{2}(\omega+\dot{\theta}_j)^2 \\
&= I\left(\omega+\dot{\theta}_j\right)\ddot{\theta}_j+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_j-\theta_k\right)+\frac{\kappa}{2}(\omega+\dot{\theta}_j)^2 \\
&\approx I\omega\ddot{\theta}_j+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_j-\theta_k\right)+\frac{\kappa}{2}\omega^2+\kappa\omega\dot{\theta}.
\end{aligned}
$$

$$
I\ddot{\theta}_j+\kappa\dot{\theta}_j=\frac{P_j}{\omega}-\frac{\kappa}{2}\omega+\sum_{k\neq j}\frac{V^2}{2\omega L}\sin\left(\theta_k-\theta_j\right).
$$

B3	The dynamical equation of θ_j is similar to that of the coupled oscillator network shown in the figure. It consists of N particles free to slide on a smooth circular rail of radius R without collision. Each particle has mass m and is subject to a force F_j for particle j in the tangential direction of the circle. When a particle moves, it experiences a damping force that is equal to minus the velocity times the damping constant b. Each pair of particles is connected by a spring of very short equilibrium length and force constant k. Derive the dynamical equation of the angular positions θ_j, and fill in the table on the answer sheet with the physical terms for the coupled oscillator network and the corresponding terms in the power grid. θ_j的动力学方程式类似于图中所示的耦合振荡器网络。它由N个可在半	**3 points** **3 分**

2

	径R的平滑圆形轨道上自由滑动而没有碰撞的粒子组成。每个粒子的质量为m。粒子j 在圆的切线方向上受到力 F_j的作用。当粒子移动时，它受到的阻尼力等于速度乘以阻尼常数b的负值。每对粒子通过平衡长度非常短和劲度系数為k的弹簧连接。导出角位置θ_j的动力学方程式，并在答题纸上的表格内填写耦合振荡器网络的物理项和电网中的相应物理项。	



Force of particle 2 on particle 1:

$$F_{12} = k2R\sin\left(\frac{\theta_2 - \theta_1}{2}\right).$$

Its tangential component is

$$F_{12t} = F_{12}\cos\left(\frac{\theta_2 - \theta_1}{2}\right) = k2R\sin\left(\frac{\theta_2 - \theta_1}{2}\right)\cos\left(\frac{\theta_2 - \theta_1}{2}\right) = kR\sin(\theta_2 - \theta_1).$$

Using Newton's law for circular motion,

$$I\frac{d^2\theta_j}{dt^2} = RF_j - Rb\frac{d\theta_j}{dt} + \sum_{k \neq j} kR\sin(\theta_k - \theta_j).$$

	Moment of inertia 转动惯量	Damping torque 阻尼力矩	Torque due to external force 由外力引起的扭矩	Torque on particle j due to particle k 粒子k作用於粒子j的扭矩

Coupled oscillator network 耦合振荡器网络	$I\dfrac{d^2\theta_j}{dt^2}$			
Corresponding term in the power grid 电网中的相应物理项	$I\dfrac{d^2\theta_j}{dt^2}$			

	Moment of inertia	Damping torque	Torque due to external force	Torque on particle j due to particle k
Coupled oscillator network	$I\dfrac{d^2\theta_j}{dt^2}$	$R^2 b\dfrac{d\theta_j}{dt}$	RF_j	$kR^2\sin(\theta_k-\theta_j)$
Corresponding term in the power grid	$I\dfrac{d^2\theta_j}{dt^2}$	$\kappa\dfrac{d\theta_j}{dt}$	$\dfrac{P_j}{\omega}-\dfrac{\kappa}{2}\omega$	$\dfrac{V^2}{2\omega^2 L}\sin(\theta_k-\theta_j)$

B4	Consider a fully connected power grid with N_c consumer nodes and N_g generator nodes, and friction is negligible. Each consumer node consumes power P and the total consumed power is evenly provided by the generator nodes. Calculate the phase difference between the generators and the consumers at the steady state. 考虑一个由N_c个功耗设备节点和N_g个发电机节点組成的完全连接电网，其中摩擦可忽略不计。每个功耗设备节点消耗功率P，并且总消耗功率由发电机节点均匀提供。计算稳定状态下发电机和功耗设备之间的相位差。	**2 points 2 分**

At the steady state,

$$\frac{P_j}{\omega}+\sum_{k\neq j}\frac{V^2}{2\omega^2 L}\sin(\theta_k-\theta_j)=0.$$

For the generators,

$$\frac{N_c P}{N_g\omega}+\frac{N_c V^2}{2\omega^2 L}\sin(\theta_c-\theta_g)=0.$$

For the consumers,

$$-\frac{P}{\omega}+\frac{N_g V^2}{2\omega^2 L}\sin(\theta_g-\theta_c)=0.$$

The two equations are dependent. Solution:

$$\theta_g-\theta_c=\arcsin\left(\frac{2\omega LP}{N_g V^2}\right).$$

B5	Calculate the minimum number of generators to keep the power grid in part (B4) synchronized. 计算在（B4）部份中的电网若要维持在同步状态，发电机数量最小需要多少部？	**1 points** **1 分**

$$\frac{2\omega LP}{N_g V^2}\leq 1,$$

Hence the minimum number of generators is

$$N_{g,\min}=\frac{2\omega LP}{V^2},$$

(or more precisely, $N_{g,\min}=$ ceiling function of $\frac{2\omega LP}{V^2}$).

Reference: F. Dörfler, M. Chertkov, and F. Bullo, Synchronization in complex networks and smart grids, Proc. Natl. Acad. Sci. USA **110**, 2005-2010 (2013).