Solution

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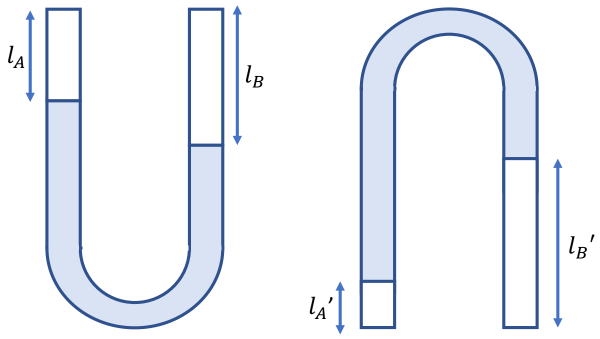
Pan Pearl River Delta Physics Olympiad 2019
2019 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:00 am – 11:30 am, 15 February, 2019)

Please fill in your final answers to all problems on the **answer sheet**.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the **answer sheet only**. Question papers and working sheets will **not** be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. A U-shaped glass tube (9 points) U 形玻璃管（9 分）

A U-shaped glass tube with a constant cross-sectional area contains mercury (with density $\rho_{Hg} = 1.36\times10^4$ kg·m^{-3}). The two ends of the tubes are sealed; one contains gas A, the other contains gas B, both of which are ideal gases. In this problem, you can take the gravitational acceleration $g = 9.8$ ms^{-2}.

具有恒定横截面积的 U 形玻璃管含有汞（具有密度$\rho_{Hg} = 1.36\times10^4$ kg·m^{-3}）。管的两端是密封的;一端含有气体 A，另一端含有气体 B，两者都是理想气体。在这个问题上，你可以采取重力加速度 $g = 9.8$ ms^{-2}。



(a) (b)

First, we set the tubes vertically, with the two ends up (Fig. 2a). The parts filled with gases A and B have lengths $l_A = 12$ cm and $l_B = 18$ cm, respectively. Then we turn the tubes upside down (Fig. 2b), the length of the parts filled by gas A and gas B are $l_A' = 6$ cm and l_B' respectively. The ambient temperature is $T = 20$°C.

首先，我们将管子垂直放置，两端朝上（图 2a）。填充有气体 A 和 B 的部份分别具有长度$l_A = 12$ cm和长度$l_B = 18$ cm。然后我们将管子倒置（图 2b），由气体 A 和气体 B 填充的部分的长度分别为$l_A' = 6$ cm和l_B'。环境温度是$T = 20$°C。

(a) Calculate the numerical value of the length l_B' of the part filled with gas B when the tube is turned upside down. [1]

(a) 计算当玻璃管倒置时填充有气体 B 的部份的长度l_B'的数值。 [1]

Solution

(b) Calculate the numerical values of the pressures p_A, p_B, p'_A and p'_B of the gases for the two orientations of the tubes. [4]
(b) 计算玻璃管内气体分别在两个方向的压强p_A，p_B，p'_A 和p'_B的数值。 [4]

(c) We look at the vertical tubes with two ends up again (Fig. 2a), but now we increase the ambient temperature by $\Delta T = 20°\mathrm{C}$ (i.e. $T \to T + \Delta T$). This leads to the changes in length $l_i \to l_i + \Delta l_i$ and pressure $p_i \to p_i + \Delta p_i$ $(i = A, B)$ for gases A and B. Determine the changes $\Delta l_A, \Delta l_B, \Delta p_A$ and Δp_B numerically. [4]
(c) 我们再看两端向上的垂直玻璃管（图 2a），但现在我们增加环境温度 $\Delta T = 20°\mathrm{C}$（即$T \to T + \Delta T$）。这导致气体 A 和 B 的长度$l_i \to l_i + \Delta l_i$和压强$p_i \to p_i + \Delta p_i (i = A, B)$发生变化。試計算长度变化 $\Delta l_A, \Delta l_B$ 和压强变化$\Delta p_A, \Delta p_B$的数值 。[4]

Solution:
(a) By assuming the volume of the mercury doesn't change, the volume filled by the gases doesn't change. We have

$$Sl_A + Sl_B = Sl'_A + Sl'_B$$
$$\Rightarrow l'_B = l_A + l_B - l'_A = 24 \text{ cm}$$

(b) By assuming the gases are ideal and the temperature doesn't change, $p_i V_i$ is conserved.

$$p_i V_i = p'_i V'_i$$
$$\Rightarrow p_A l_A = p'_A l_A{}' \text{ and } p_B l_B = p'_B l_B{}'$$

In the case of the vertical tube (Fig. 2a), the pressure difference between two parts is

$$p_A + \rho_{Hg} g(l_B - l_A) = p_B$$
$$\Rightarrow p_B - p_A = \rho_{Hg} g(l_B - l_A)$$

Similarly, in the configuration (Fig.2b),

$$p'_B + \rho_{Hg} g(l'_B - l_A{}') = p'_A$$
$$\Rightarrow p'_A - p'_B = \rho_{Hg} g(l'_B - l'_A)$$

Now we have 4 equations and 4 unknowns, we can solve them uniquely.

$$p_B = p_A + \rho_{Hg} g(l_B - l_A)$$
$$\Rightarrow p_A = \rho_{Hg} g \left(l'_B - l'_A + \frac{l_B}{l_B{}'}(l_B - l_A) \right) \left(\frac{l_A}{l_A{}'} - \frac{l_B}{l_B{}'} \right)^{-1}$$

We get

$$p_A = 24 \text{ kPa}$$
$$p_B = 32 \text{ kPa}$$
$$p_A{}' = 48 \text{ kPa}$$
$$p_B{}' = 24 \text{ kPa}$$

(c) (Method 1: Keep 1st order term) By increasing the temperature, all variables are changing accordingly,

$$T \to T + \Delta T$$
$$p_i \to p_i + \Delta p_i$$
$$l_i \to l_i + \Delta l_i$$

The ideal gas law becomes,

Solution

$$(p_i + \Delta p_i)(l_i + \Delta l_i)S = n_i R(T + \Delta T)$$
$$\Rightarrow p_i l_i S + (p_i \Delta l_i + \Delta p_i l_i)S + \cancel{\Delta p_i \Delta l_i S} = n_i RT + n_i R\Delta T$$

By dividing $n_i R = \frac{p_i l_i S}{T}$, we get

$$\frac{\Delta l_i}{l_i} + \frac{\Delta p_i}{p_i} = \frac{\Delta T}{T} \quad (i = A, B)$$

Next, we consider the length variation,

$$(p_B + \Delta p_B) - (p_A + \Delta p_A) = \rho_{Hg} g\big((l_B + \Delta l_B) - (l_A + \Delta l_A)\big)$$
$$\Rightarrow (p_B - p_A) + (\Delta p_B - \Delta p_A) = \rho_{Hg} g(l_B - l_A) + \rho_{Hg} g(\Delta l_B - \Delta l_A)$$
$$\Rightarrow (\Delta p_B - \Delta p_A) = \rho_{Hg} g(\Delta l_B - \Delta l_A)$$

Finally, we assume the volume of the mercury doesn't change,

$$\Delta l_A + \Delta l_B = 0$$

We get 4 equations and we can solve for 4 unknowns Δp_i and Δl_i.

$$\Rightarrow \Delta p_i = \left(\frac{\Delta T}{T} - \frac{\Delta l_i}{l_i}\right) p_i$$
$$\Rightarrow \Delta l_A = -\Delta l_B$$

$$\left(\frac{\Delta T}{T} - \frac{\Delta l_B}{l_B}\right) p_B - \left(\frac{\Delta T}{T} + \frac{\Delta l_B}{l_A}\right) p_A = \rho_{Hg} g(\Delta l_B + \Delta l_B)$$
$$\Rightarrow \frac{\Delta T}{T}(p_B - p_A) = \Delta l_B \left(2\rho_{Hg} g + \frac{p_B}{l_B} + \frac{p_A}{l_A}\right)$$
$$\Rightarrow \Delta l_B = \frac{\Delta T(p_B - p_A)}{T\left(2\rho_{Hg} g + \frac{p_B}{l_B} + \frac{p_A}{l_A}\right)}$$

The numerical values is

$$\Delta l_B = 0.085 \text{ cm}$$
$$\Delta l_A = -\Delta l_B = -0.085 \text{ cm}$$

$$\Delta p_B = 2033 \text{ Pa}$$
$$\Delta p_A = 1808 \text{ Pa}$$

(Method 2: Exact. Keep cubic term) We can retain the 2nd order of the equation,

$$(p_i \Delta l_i + \Delta p_i l_i)S + \Delta p_i \Delta l_i S = n_i R\Delta T$$
$$\Rightarrow \frac{\Delta l_i}{l_i} + \frac{\Delta p_i}{p_i} + \frac{\Delta p_i}{p_i}\frac{\Delta l_i}{l_i} = \frac{\Delta T}{T}$$

Together with the equations,

$$\Delta l_A + \Delta l_B = 0$$
$$(\Delta p_B - \Delta p_A) = \rho_{Hg} g(\Delta l_B - \Delta l_A) = 2\rho_{Hg} g \Delta l_B \qquad (A)$$
$$\frac{\Delta p_A}{p_A}\left(1 - \frac{\Delta l_B}{l_A}\right) = \frac{\Delta T}{T} + \frac{\Delta l_B}{l_A}$$
$$\frac{\Delta p_B}{p_B}\left(1 + \frac{\Delta l_B}{l_B}\right) = \frac{\Delta T}{T} - \frac{\Delta l_B}{l_B}$$

Sub. Into Eqtn. (A),

Solution

$$\Rightarrow\left(\frac{\frac{\Delta T}{T}-\frac{\Delta l_B}{l_B}}{1+\frac{\Delta l_B}{l_B}}\right)p_B-\left(\frac{\frac{\Delta T}{T}+\frac{\Delta l_B}{l_A}}{1-\frac{\Delta l_B}{l_A}}\right)p_A=2\rho_{Hg}g\Delta l_B$$

Numerically, we get $\Delta l_B = 0.000814$ m. (The other two roots are larger than l_B and hence we neglected.

$$\Delta l_A=-\Delta l_B=-0.000814\,m$$

Accordingly, we get

$$\Delta p_A = 1813\text{ Pa}$$
$$\Delta p_B = 2030\text{ Pa}$$

(Method 3: Keep 2nd order term) When the temperature increases from T to $T_1 = T + \Delta T$, Eq. (1) becomes

$$\frac{l_B}{l_B^1}p_B-\frac{l_A}{l_A^1}p_A=\rho g h_1\frac{T}{T_1}.$$

Let $l_A^1 = l_A + x$. Then $l_B^1 = l_B - x$ and $h_1 = l_B^1 - l_A^1 = (l_A - l_B) - 2x = h - 2x$. The equation becomes

$$\frac{p_B l_B}{l_B-x}-\frac{p_A l_A}{l_A+x}=\rho g(h-2x)\frac{T}{T_1}.$$

Expanding and keeping only up to quadratic terms of x,

$$p_B l_B(l_A+x)-p_A l_A(l_B-x)=\rho g\frac{T}{T_1}(h-2x)(l_B-x)(l_A+x).$$

$$(p_B-p_A)l_A l_B+(p_A l_A+p_B l_B)x$$
$$=\rho g\frac{T}{T_1}[hl_A l_B+(hl_B-hl_A-2l_A l_B)x+(2l_A-2l_B-h)x^2].$$

$$6\rho g h^3+18\rho g h^2 x=\rho g\frac{T}{T_1}[6h^3-11h^2x-3hx^2].$$

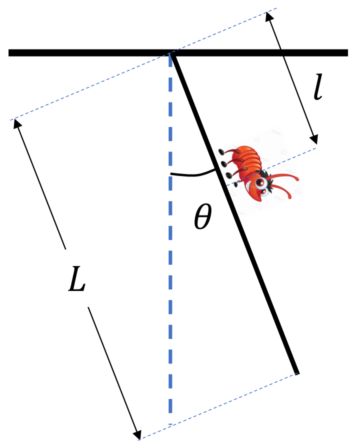
$$\left(\frac{T_1}{T}-1\right)2h^2+\left(6\frac{T_1}{T}+\frac{11}{3}\right)hx+x^2=0.$$

$$x=-0.0814\text{ cm}.$$

which is the same as the exact solution.

Solution

2. Bug on a rod (11 points) 杆上的虫子（11 分）



A pendulum consists of a uniform rigid rod of length L, mass M, a bug of mass $M/3$ which can crawl along the rod. The rod is pivoted at one end and swings in a vertical plane. Initially the bug is at the pivot-end of the rod, which is at rest at an angle θ_i ($\theta_i \ll 1$ rad) from the vertical as shown in the figure, is released. For $t > 0$ the bug crawls slowly with constant speed V along the rod towards the bottom end of the rod.

单摆由长度L、质量M的均匀刚性杆组成， 一只质量为$M/3$的虫子沿着杆爬行。杆在一端枢转并在垂直平面中摆动。虫子最初位于杆的枢轴处 ，此时杆与垂直方向成一定夾角θ_i ($\theta_i \ll 1$ rad) (如图所示)。當$t > 0$， 虫子沿着杆朝着杆的底端以恒定速度V缓慢地爬行。

(a) What is the moment of inertia I of the rod and bug about the pivot when the bug has reached a distance l along the rod. [1]

(a) 当虫子沿着杆爬至距离 l 时，杆和虫子相對於枢轴的转动慣量 I 是什么? [1]

(b) Find the angular frequency ω of the swing of the pendulum when the bug has reached a distance l along the rod. Express your answer in terms of L and l. [1]

(b) 当虫子沿着杆爬至距离 l 时，找出单摆摆动的角频率ω。答案用L和l表达 。 [1]

From now on, you can assume the speed of the bug is so small that l hardly changes in a period of oscillation and can be taken to be constant, and the motion of the bug can be effectively described by simple harmonic motion, that is,

从现在开始，您可以假设虫子的速率非常小，因此在振荡周期内几乎不会发生变化，并可当为常数 ，而虫子的运动可以用简谐运动描述，即是

$$\theta(t) = \theta_0(l)\sin\omega t$$

where ω is the angular frequency you obtained in part (b) and $\theta_0(l)$ is the amplitude of the oscillation which will vary as the bug crawled.

其中 ω 是你在（b）部份中得到的角频率，$\theta_0(l)$ 是振荡的幅度，并随着虫子爬行而变化。

After the bug has reached a distance l along the rod, calculate the following quantities when it further crawls a short distance Δl. In parts (c) to (f), express your answers in terms of Δl, $\Delta\theta_0$ and other parameters in the problem, where $\Delta\theta_0$ is the change of the angular amplitude during the displacement Δl.

试计算在虫子沿着杆子爬至距离 l后，当虫子继续爬行短距离Δl时下列的物理量。在(c)至(f)部份中，答案以 Δl、$\Delta\theta_0$及本题中的其他参数表达，其中$\Delta\theta_0$是振荡幅度在位移Δl后的改变量。

(c) Calculate the time-averaged work done ΔW by the bug on the rod-bug system. [1]

(c) 试计算虫子作用于杆子-虫子系统的时间平均功ΔW. [1]

Solution

(d) Calculate the change $\Delta\omega^2$ in the term ω^2. [1]
(d) 试计算ω^2 的改变量$\Delta\omega^2$。[1]

(e) Calculate the time-averaged change ΔK in the kinetic energy of the whole system. [1]
(e) 试计算整个系统的时间平均动能改变量ΔK。[1]

(f) Calculate the time-averaged change ΔU in the potential energy of the whole system. [1]
(f) 试计算整个系统的时间平均势能改变量ΔU。[1]

(g) Summarizing the above steps, calculate the relation between $\Delta\theta_0$ and Δl. [3]
(g) 总结上述步骤，计算$\Delta\theta_0$和Δl之间的关系。

(h) Find the amplitude of the swing of the pendulum when the bug reaches the bottom end of the rod ($l = L$). Express your answer in terms of $\theta_0(l = 0) = \theta_i$. [2]
(h) 当虫子爬至杆的底端时($l = L$)，找出单摆摆动的幅度。答案用 $\theta_0(l = 0) = \theta_i$ 表达。[2]

Solution:
(a) When the bug has crawled at distance l, the moment of inertia of the rod and bug along the pivot is

$$I = \frac{1}{3}ML^2 + \frac{1}{3}Ml^2 = \frac{1}{3}M(L^2 + l^2)$$

(b) The equation of motion of the pendulum is

$$\frac{d}{dt}\left(I\dot{\theta}\right) = -Mg\frac{L}{2}\sin\theta - \frac{1}{3}Mgl\sin\theta$$

$$\Rightarrow \frac{1}{3}M(L^2 + l^2)\ddot{\theta} + \frac{2}{3}Ml\dot{l}\dot{\theta} = -Mg\sin\theta\left(\frac{L}{2} + \frac{l}{3}\right)$$

For small oscillations, it becomes:

$$\ddot{\theta} + \frac{2l\dot{l}\dot{\theta}}{L^2 + l^2} + \frac{g\left(l + \frac{3L}{2}\right)\theta}{L^2 + l^2} = 0$$

If the bug crawls so slowly that the change in l in a period of oscillation is negliglbe, i.e. $\dot{l} = v \ll l\omega$, we can ignore the second term and write,

$$\ddot{\theta} + \frac{g\left(l + \frac{3L}{2}\right)\theta}{L^2 + l^2} = 0$$

Hence the angular frequency of oscillation is

$$\omega = \sqrt{\frac{g(3L + 2l)}{2(L^2 + l^2)}}$$

(c) Consider the motion of the bug along the rod,

$$\frac{M}{3}\left(\ddot{l} - l\dot{\theta}^2\right) = \frac{Mg\cos\theta}{3} - f$$

when f is the force exerted on the bug by the rod. As the bug crawls with constant speed, $\ddot{l} = 0$. We get

$$f = \frac{Mg}{3}\cos\theta + \frac{Ml\dot{\theta}^2}{3}$$

Solution

The frictional force is pointing towards the pivot. This force is exerted by the bug on the rod-bug system to maintain its slow motion. Hence

$$\Delta W = -f\Delta l = -\frac{M}{3}\left(g\cos\theta + l\dot{\theta}^2\right)\Delta l \approx \left(-\frac{Mg}{3} + \frac{Mg}{6}\theta^2 - \frac{Ml}{3}\dot{\theta}^2\right)\Delta l.$$

Since the bug moves slowly along the rod, the motion can be approximated by simple harmonic motion. Averaging over time,

$$\langle\theta^2\rangle = \theta_0^2\langle\sin^2\omega t\rangle = \frac{1}{2}\theta_0^2,$$

$$\langle\dot{\theta}^2\rangle = \theta_0^2\omega^2\langle\cos^2\omega t\rangle = \frac{1}{2}\omega^2\theta_0^2.$$

Hence

$$\Delta W \approx \left(-\frac{Mg}{3} + \frac{Mg}{12}\theta_0^2 - \frac{Ml}{6}\omega^2\theta_0^2\right)\Delta l.$$

(d)

$$\frac{d\omega^2}{dl} = \frac{d}{dl}\left[\frac{g(3L+2l)}{2(L^2+l^2)}\right] = g\left[\frac{1}{L^2+l^2} - \frac{l(3L+2l)}{(L^2+l^2)^2}\right].$$

$$\Delta\omega^2 = g\left[\frac{1}{L^2+l^2} - \frac{l(3L+2l)}{(L^2+l^2)^2}\right]\Delta l.$$

(e) The kinetic energy is

$$K = \frac{1}{2}\frac{M}{3}(L^2+l^2)\dot{\theta}^2 + \frac{1}{2}\frac{M}{3}V^2.$$

Averaging over time,

$$K \approx \frac{M}{12}(L^2+l^2)\omega^2\theta_0^2 + \frac{M}{6}V^2.$$

Hence

$$\Delta K \approx \frac{M}{6}(L^2+l^2)\omega^2\theta_0\Delta\theta_0 + \frac{M}{6}l\omega^2\theta_0^2\Delta l + \frac{M}{12}(L^2+l^2)\theta_0^2\Delta\omega^2$$
$$= \frac{M}{6}(L^2+l^2)\omega^2\theta_0\Delta\theta_0 + \frac{M}{6}l\omega^2\theta_0^2\Delta l + \frac{Mg}{12}\theta_0^2\left[1 - \frac{l(3L+2l)}{L^2+l^2}\right]\Delta l.$$

Substituting the expression of ω^2,

$$\Delta K \approx \frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0 + \frac{Mg}{12}\theta_0^2\Delta l.$$

(f) The potential energy is

$$U = -\frac{MgL}{2}\cos\theta - \frac{Mgl}{3}\cos\theta \approx -\frac{Mg}{6}(3L+2l) + \frac{Mg}{12}(3L+2l)\theta^2.$$

Averaging over time,

$$U \approx -\frac{Mg}{6}(3L+2l) + \frac{Mg}{24}(3L+2l)\theta_0^2.$$

Hence

$$\Delta U \approx -\frac{Mg}{3}\Delta l + \frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0 + \frac{Mg}{12}\theta_0^2\Delta l.$$

(g) Using the work-energy theorem,

$$\Delta W = \Delta K + \Delta U.$$

$$\left(-\frac{Mg}{3} + \frac{Mg}{12}\theta_0^2 - \frac{Ml}{6}\omega^2\theta_0^2\right)\Delta l \approx \frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0 + \frac{Mg}{12}\theta_0^2\Delta l$$

$$-\frac{Mg}{3}\Delta l+\frac{Mg}{12}(3L+2l)\theta_0\Delta\theta_0+\frac{Mg}{12}\theta_0^2\Delta l.$$

Simplifying,

$$\frac{\Delta\theta_0}{\theta_0}+\frac{1}{2}\left(\frac{1}{3L+2l}+\frac{l}{L^2+l^2}\right)\Delta l\approx 0.$$

$$\Delta\theta_0\approx-\frac{\theta_0}{2}\left(\frac{1}{3L+2l}+\frac{l}{L^2+l^2}\right)\Delta l.$$

(h) Integrating,

$$\ln\theta_0+\frac{1}{4}\ln(3L+2l)+\frac{1}{4}\ln(L^2+l^2)\approx \text{constant}.$$

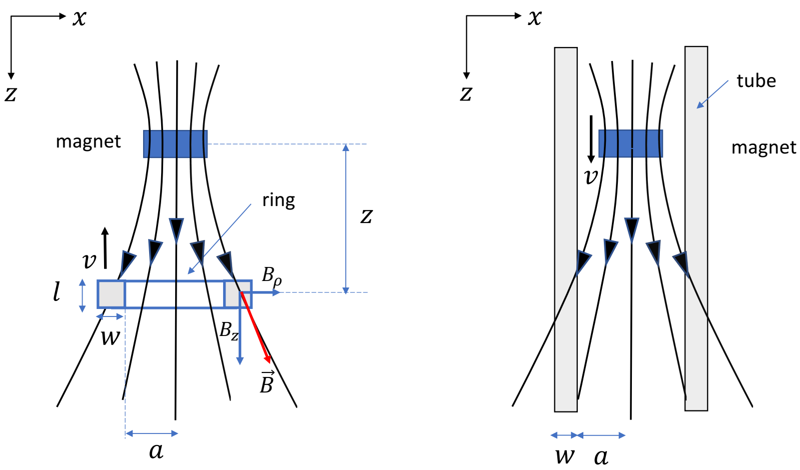
$$\theta_0(3L+2l)^{\frac{1}{4}}(L^2+l^2)^{1/4}\approx \text{constant}.$$

$$\theta_0(l)\approx\left(\frac{3}{10}\right)^{\frac{1}{4}}\theta_0(l=0)=0.74\,\theta_i.$$

3. Falling magnet inside a conductive pipe (10 points) 在导电管内下落的磁铁（10 分）

In this question, we consider the motion of a strong tiny magnet with mass M and magnetic dipole moment μ falling inside a vertical conducting non-magnetic tube.

在这个问题中，我们考虑具有质量M和磁偶极矩μ的强磁铁落入垂直的导电非磁性管内的运动。



(a) (b)

(a) We first consider a ring (with radius a, length l, thickness w ($w \ll a$) and conductivity σ) moving towards the magnet with speed v as shown in figure (a). The magnetic field at position $\vec{r}$ due to a magnetic dipole $\vec{\mu}=\mu\hat{z}$ (pointing downward as positive) at origin is given by

我们首先考虑一个环（半径a，长度l，厚度w（$w \ll a$）和导电率σ）以速度v向磁铁移动，如图（a）所示。由位於原點的磁偶极子$\vec{\mu}=\mu\hat{z}$（指向下为正）所產生的磁场，在位置$\vec{r}$处为

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^2}-\vec{\mu}\right)$$

Solution

1

(i) Calculate the components of the magnetic field $B_\rho(\vec{r})$ and $B_z(\vec{r})$ at the point $\vec{r}$ with cylindrical coordinate $\vec{r} = (\rho = a, \phi = 0, z)$ [2]
(i) 计算磁场在位置$\vec{r}$处的分量$B_\rho(\vec{r})$和$B_z(\vec{r})$，$\vec{r}$的圆柱坐标为$\vec{r} = (\rho = a, \phi = 0, z)$。[2]

(ii) Calculate the induced emf and current on the ring. [3]
(ii) 计算环上的感生电动势和电流。 [3]

(iii) Calculate the magnetic force experienced by the coil. [1]
(iii) 计算环所感受的磁力。 [1]

(b) Next we consider a magnet falling inside along a vertical conducting non-magnetic tube of infinite length (with radius a, thickness w and conductivity σ) as shown in figure (b).
(b) 接下来我们考虑沿着一个无限长度（具有半径a，厚度w和导电率σ）的垂直导电非磁性管内落下的磁铁，如图（b）所示。

(i) When the magnet falls with the speed v, it experiences a damping force with the magnitude equal to γv. Calculate the damping constant γ. [3]
(i) 当磁铁以速度v下降时，它会感受一个量值为γv的阻尼力。试计算阻尼常数γ。 [3]

Hints: The following mathematical identity may be useful.
提示：下列数学公式可能有用。

$$\int_{-\infty}^{\infty} \frac{u^2 du}{(1+u^2)^5} = \frac{5\pi}{128}$$

(ii) Determine the terminal velocity of the magnet when it falls inside the tube. [1]
(ii) 試計算磁铁在管内落下时的终端速度。 [1]

Solution:
(ai) We can use the cylindrical coordinate where

$$\vec{r} = z\hat{z} + a\hat{\rho}$$

Hence

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{(z^2+a^2)^{3/2}} \left(\frac{3z\mu}{z^2+a^2}(z\hat{z} + a\hat{\rho}) - \mu\hat{z} \right)$$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 \mu}{4\pi} \frac{1}{(z^2+a^2)^{5/2}} \left((2z^2 - a^2)\hat{z} + 3az\hat{\rho} \right)$$

$$B_z = \frac{\mu_0 \mu}{4\pi} \frac{(2z^2 - a^2)}{(a^2+z^2)^{5/2}}$$

$$B_\rho = \frac{\mu_0 \mu}{4\pi} \frac{3za}{(a^2+z^2)^{5/2}}$$

(aii) The induced emf is given by

$$\epsilon = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = 2\pi a v B_\rho = \frac{\mu_0 \mu v}{2} \frac{3za^2}{(a^2+z^2)^{5/2}}$$

Alternatively, the flux through the ring is given by

Solution

$$\Phi = \int_0^a 2\pi\rho d\rho B_z(\rho, z) = \frac{\mu_0\mu}{4}\int_0^a 2\rho d\rho \left[\frac{3z^2}{(\rho^2+z^2)^{\frac{5}{2}}} - \frac{1}{(\rho^2+z^2)^{\frac{3}{2}}}\right]$$

$$= \frac{\mu_0\mu}{4}\left[-\frac{2z^2}{(\rho^2+z^2)^{\frac{3}{2}}} + \frac{2}{(\rho^2+z^2)^{\frac{1}{2}}}\right]_0^a = \frac{\mu_0\mu}{2}\frac{a^2}{(a^2+z^2)^{\frac{3}{2}}}.$$

The induced emf is given by

$$\epsilon = -\frac{d\Phi}{dt} = \frac{\mu_0\mu v}{2}\frac{3a^2z}{(a^2+z^2)^{\frac{5}{2}}}$$

and the resistance of the coil is

$$R = \frac{2\pi a}{\sigma w l}$$

Hence the induced current is

$$I = \frac{\epsilon}{R} = \frac{2\pi a v B_\rho}{2\pi a}\sigma w l = v B_\rho \sigma w l = \frac{\mu_0\mu v\sigma w l}{4\pi}\frac{3za}{(a^2+z^2)^{5/2}}$$

(aiii) The magnetic force experienced by the coil is

$$F = I(2\pi a)B_\rho = (2\pi a l w)\sigma v B_\rho^2$$

(bi) The total magnetic force experienced by the magnet is

$$F = \int dF = (2\pi a^3 w\sigma v)\left(\frac{3\mu_0\mu}{4\pi}\right)^2 \int_{-\infty}^{\infty}\frac{z^2 dz}{(a^2+z^2)^5}$$

$$\Rightarrow F = \frac{9\mu_0^2\mu^2 w\sigma v}{8\pi a^4} \times \int_{\infty}^{\infty}\frac{u^2 du}{(1+u^2)^5} = \frac{45}{1024}\left(\frac{\mu_0^2\mu^2 w\sigma}{a^4}\right)v = \gamma v$$

(bii) Newton's 2nd law gives

$$mg - \gamma\frac{dz}{dt} = m\frac{d^2z}{dt^2}$$

At the terminal velocity, $\frac{d^2z}{dt^2} = 0$ and

$$v_t = \frac{mg}{\gamma} = \frac{1024}{45}\left(\frac{a^4 g}{\mu_0\mu^2 w\sigma}\right)$$

4. Heat flux between two plates (10 points) 两块板之间的热通量（10 点）

A system composed of two parallel plates at distance L from each other, which are at temperature T_1 and T_2 respectively.
一个系统由两块互相分隔、距离為 L 的平行板组成，分别处于温度 T_1 和 T_2。

(a) Calculate the heat flux density P (i.e. rate of heat energy flow per unit area) between two plates if the space between the plates is vacuum and each of the plates has emissivity ϵ. [4]
(a) 如果两块板之间的空间是真空并且每个板具有比輻射率 ϵ，计算两块板之间的热通量密度 P（即每单位面积的热能流速率）。[4]

Solution

(b) Now the space between the plates is filled with a monoatomic gas of molar density n and molar mass M. You need to estimate the heat flux density P between two plates according to the following approximations:

- The gas density is so low that the mean free path $\lambda \gg L$.
- $T_1 \gg T_2$
- When gas molecules bounce from the plates, they obtain the temperature of the respective plates (for instance, if they are absorbed/bounded for a short time by the molecules of the plate, and then released back into the space between the plates).
- You may neglect the black body radiation.
- "Estimate" means that the numeric prefactor of your expression does not need to be accurate.

(b) 现在两块板之间的空间充满了摩尔密度 n 和摩尔质量 M 的单原子气体。您需要跟据以下近似估算两块板之间的热通量密度 P：

- 气体密度低至平均自由程 $\lambda \gg L$.
- $T_1 \gg T_2$
- 当气体分子从板上反弹时，它们会获得相应板的温度（例如，如果它们被板的分子吸收/束缚很短的时间，然后释放回板之间的空间）。
- 您可忽略黑体辐射。
- “估算”表示答案的数字前因子不需要准确。

(i) Consider that there is an atom colliding with the hot plate and remains in thermal equilibrium with the hot plate when it is reflected by the plate. Calculate the average velocity square $\langle v_1^2 \rangle$, and estimate the average horizontal velocity $\langle v_{1x} \rangle$ of the atom. [0.5]
(i) 考虑一粒原子与热板碰撞并且当它被板反射时与热板保持热平衡。计算平均速度平方$\langle v_1^2 \rangle$，并估算原子的平均水平速度$\langle v_{1x} \rangle$。 [0.5]

(ii) Consider that there is an atom colliding with the cold plate and remains in thermal equilibrium with the cold plate when it is reflected by the plate. Calculate the average velocity square $\langle v_2^2 \rangle$ and estimate the average horizontal velocity $\langle v_{2x} \rangle$ of the atom. [0.5]
(ii) 考虑一粒原子与冷板碰撞并且当它被板反射时与冷板保持热平衡。计算平均速度平方$\langle v_2^2 \rangle$，并估算原子的平均水平速度$\langle v_{2x} \rangle$。 [0.5]

(iii) Find the average energy transmitted by an atom when it moves from the hot to the cold plate. [1]
(iii) 找出一个原子从热板移动到冷板时传输的平均能量。 [1]

(iv) Estimate the heat flux density P between two plates. [4]
(iv) 估算两块板之间的热通量密度 P。 [4]

Solution:
(a) Without considering reflection of heat, it is reasonable to estimate the heat flux density

$$P \approx \sigma\epsilon(T_1^4 - T_2^4)$$

(N.B. You will receive 1 points for this answer)

To get the exact result, we consider the recursive reflection of heat flux between two walls.

Solution

Let $Q_2 = A\sigma\epsilon T_2^4$ be the initial inward flux and $Q_1 = A\sigma\epsilon T_1^4$ be the initial outward flux without any reflection of heat.

Surface 1 emits	Q_1
Surface 2 absorbs	ϵQ_1
Surface 2 reflects	$(1-\epsilon)Q_1$
Surface 1 absorbs	$(1-\epsilon)\epsilon Q_1$
Surface 1 reflects	$(1-\epsilon)^2 Q_1$
Surface 2 absorbs	$(1-\epsilon)^2\epsilon Q_1$
Surface 2 reflects	$(1-\epsilon)^3 Q_1$
Surface 1 absorbs	$(1-\epsilon)^3\epsilon Q_1$

The same is true for the heat emitted from surface 2.

Hence the total heat flux radiated from surface 1 and reabsorbed by surface 1 is

$$\tilde{Q}_1 = (1-\epsilon)\epsilon Q_1 + (1-\epsilon)^3\epsilon Q_1 + (1-\epsilon)^5\epsilon Q_1 + \cdots$$
$$= (1-\epsilon)\epsilon Q_1(1 + (1-\epsilon)^2 + (1-\epsilon)^4 + \cdots) = \frac{(1-\epsilon)\epsilon Q_1}{1-(1-\epsilon)^2}$$

Similarly, the total heat flux radiated from surface 2 and reabsorbed by surface 2 is

$$\tilde{Q}_2 = \frac{(1-\epsilon)\epsilon Q_2}{1-(1-\epsilon)^2}$$

And hence the total heat flux radiated from surface 2 and absorbed by surface 1 is

$$Q_2 - \tilde{Q}_2 = \frac{1-(1-\epsilon)^2-(1-\epsilon)\epsilon}{1-(1-\epsilon)^2}Q_2 = \frac{\epsilon}{1-(1-\epsilon)^2}Q_2$$

The net heat flux from surface 1 to surface 2 is

$$P = Q_1 - \frac{(1-\epsilon)\epsilon Q_1}{1-(1-\epsilon)^2} - \frac{\epsilon}{1-(1-\epsilon)^2}Q_2 = \frac{\sigma\epsilon(T_1^4 - T_2^4)}{2-\epsilon}$$

(N.B. When $\epsilon = 1$, we have ideal blackbody. The heat will be completely absorbed by the plates and two results are identical.)

(b)
(i) From the kinetic theory of ideal gas, we know

$$\frac{1}{2}m\overline{v_1^2} = \frac{3}{2}kT_1 \Rightarrow \overline{v_1^2} = \frac{3kT_1}{m}$$

As an approximation,

Solution

$$v_{1x} \approx \sqrt{\overline{v_{1x}^2}} = \sqrt{\frac{1}{3}\overline{v_1^2}} = \sqrt{\frac{kT_1}{m}}$$

Remark: To be precise, we can calculate $\bar{v}_{1x}$ using the Maxwell distribution. Since the particle can only move in one direction, the Maxwell distribution becomes $g(v_{1x})dv_{1x} = 2\sqrt{\frac{m}{2\pi kT_1}}\exp\left(-\frac{mv_{1x}^2}{2kT_1}\right)dv_{1x}$ and the mean value of v_{1x} is

$$\overline{v_{1x}} = \int_0^\infty 2\sqrt{\frac{m}{2\pi kT_1}}\exp\left(-\frac{mv_{1x}^2}{2kT_1}\right)v_{1x}dv_{1x} = \sqrt{\frac{2}{\pi}}\times\sqrt{\frac{kT_1}{m}}$$

which is different by a factor of order 1.

(ii) Similarly we have

$$\overline{v_2^2} = \frac{3kT_2}{m}$$

$$v_{2x} \approx \sqrt{\overline{v_{2x}^2}} = \sqrt{\frac{1}{3}\overline{v_2^2}} = \sqrt{\frac{kT_2}{m}}$$

(iii) Since $\lambda \gg L$, the probability of collision of two molecules is ignorably small, so we can imagine the molecules as independent ones bouncing back and forth between the plates. We are now considering a two-way journey of a molecule. Let the velocity of a molecule when leaving the hot plate be v_1, while that when leave the cold plate v_2; the component perpendicular to the plates is v_{1x} and v_{2x} respectively.

The net transmitted energy is $\Delta E = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{3k}{2}(T_1 - T_2) \approx \frac{3k}{2}T_1$

(iv) The time required to cover the normal distance L back and forth is

$$\Delta t = \frac{L}{v_{1x}} + \frac{L}{v_{2x}}.$$

Since $T_1 \gg T_2$, the velocities satisfy $v_1 \gg v_2$ and

$$\Delta E \approx \frac{1}{2}mv_1^2 \text{ and } \Delta t \approx \frac{L}{v_{2x}}$$

The power transmitted (per atom) during this journey is

$$P = \frac{\Delta E}{\Delta t} = \frac{m}{2L}v_1^2 v_{2x}$$

And the heat flux density due to all particles is

$$P_{tot} = \frac{m}{2L}v_1^2 v_{2x}\times(nN_A AL)$$

where A is the surface area of the plate. The heat flux density is

$$\Rightarrow P \approx \frac{nN_A}{2\sqrt{m}}k^{\frac{3}{2}}T_1 T_2^{1/2}$$

Since $N_A m = M$ and $kN_A = R$, we have

1

Solution

$$P = \frac{n}{2}\frac{R^{\frac{3}{2}}}{\sqrt{M}}T_1\sqrt{T_2}$$

N.B. Any answer of the form

$$P = C\frac{nR^{\frac{3}{2}}}{\sqrt{M}}T_1\sqrt{T_2}$$

for some dimensionless number C (of order one) will receive full credits.