# Pan Pearl River Delta Physics Olympiad 2020
# 2020 年泛珠三角及中华名校物理奥林匹克邀请赛
# Sponsored by Institute for Advanced Study, HKUST
# 香港科技大学高等研究院赞助

## Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
## 简体版卷-2（共2题，60分）

## (1:30 pm – 5:00 pm, 8 August 2020)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

# Problem 1: Precision measurement of the gravitational constant *G* (27 points)

# 问题 1：重力常数 G 的精确测量（27 分）

The precision measurement of the gravitational constant $G$ is important because it is a fundamental constant. Besides, it can play a role in verifying (or disproving) some recent proposed versions of string theory or the existence of the fifth fundamental force.

引力常数 $G$ 是一个基本物理常数，所以对其精确测量非常重要。此外，引力常数的精确测量也有助于验证（或排除）一些弦理论的版本或第五种基本力的存在。

## Part A. Estimation of the Gravitational Field Change During the Experiment (15 points)
## 对实验中引力场变化的估计 （15 分）

| | | |
|---|---|---|
| A1 | Let $R$ and $M$ be the radius and the mass of the Earth, respectively, express the gravitational field $g$ in terms of $R$, $M$, and $G$, while ignoring the spinning of the Earth.<br>设 $R$ 和 $M$ 分别是地球的半径和质量。忽略地球自转，求引力场的强度 $g$，用 $R$，$M$ 和 $G$ 表示。 | 1 point<br>1 分 |

$$g = \frac{GM}{R^2}.$$

In this problem we present a simplified version of the latest method, which can lower the relative error of $G$ down to $10^{-5}$. As shown in Fig. 1(a), laser interferometer 1 measures the spacing between the two pendulum bobs with respect to the reference spacing between the suspension points of the pendulum, which is measured by laser interferometer 2. When the four source masses are moved from the outer position (shown in Fig. 1(a)) to the inner position (shown in Fig. 1(b)), the pendulum bob separation changes. Not pictured is the vacuum chamber that encloses the pendulums but not the source masses.

本题简化地讨论一个测量引力常数的最新实验，可以将 $G$ 的相对误差降低到 $10^{-5}$。如图 1(a) 所示，激光干涉仪 1 测量两个单摆摆锤之间的距离。激光干涉仪 2 测量两个单摆悬挂点之间的距离作为参照。当四个质量源从外侧位置（如图 1(a)所示）移动到内侧位置（如图 1(b)所示）时，两个摆锤之间的距离会改变。除了质量源外，整个实验装置置于真空环境内（没有画在图中）。

**2**

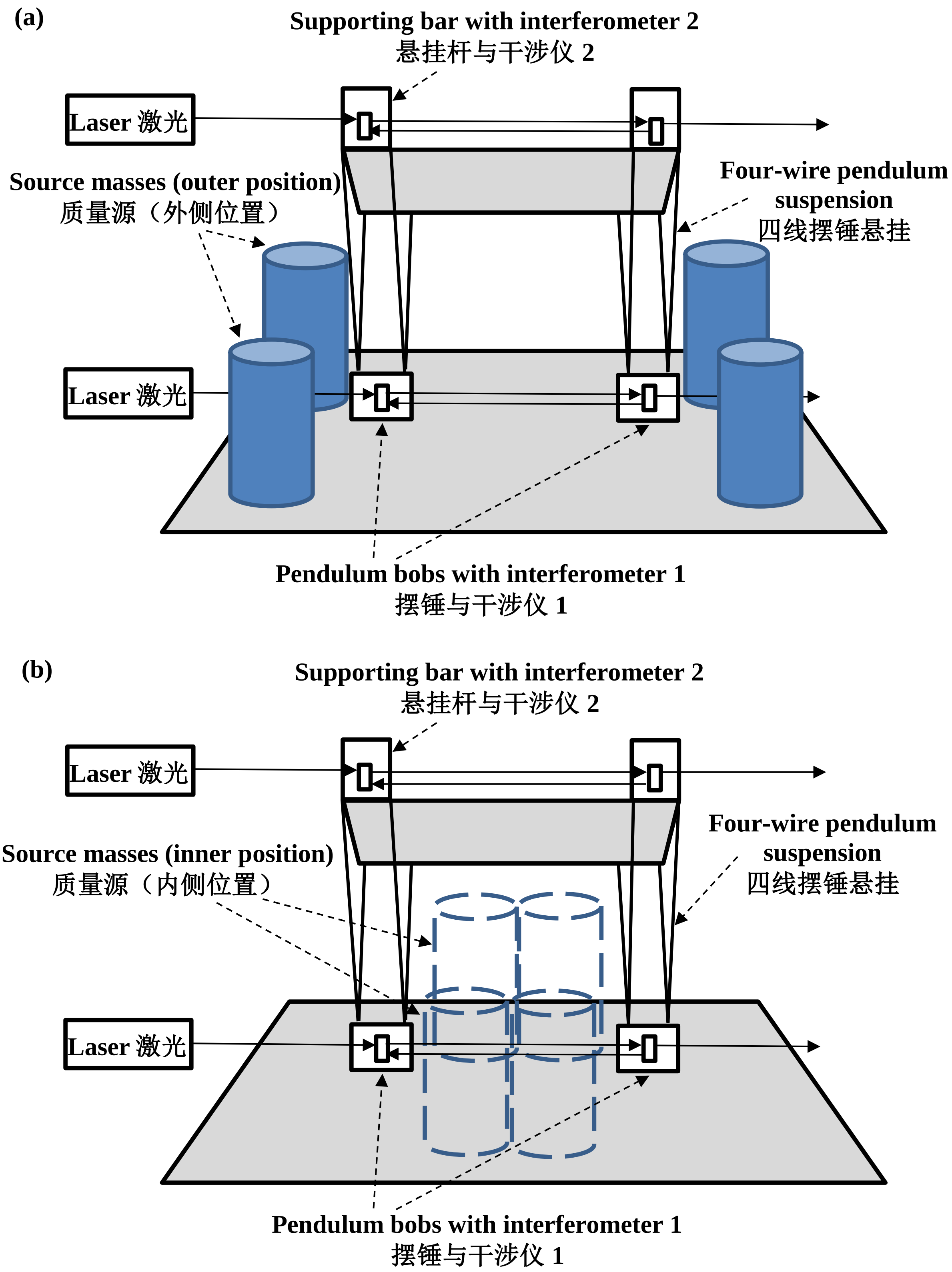


*Figure 1: The setup of apparatus for measuring the gravitational constant precisely, with the source masses placed at (a) outer positions, (b) inner positions. 引力常数精确测量的实验装置，质量源置于 (a) 外侧位置，(b)内侧位置。*

Figures 2 and 3 show the top and side views of the apparatus. The outer and inner positions of the source masses, and the pendulum bobs (at the ends of interferometer 1) are located symmetrically with respect to the center of the vacuum system. The length scales $a_1$, $a_2$, $b$, $d$, $h$ and $R$ are shown in the figures.

图 2 和图 3 是实验装置的顶视图和侧视图。质量源的外侧和内侧位置，和摆锤（在干涉仪 1 的末端）的位置相对于真空系统的中心是对称的。图中显示距离 $a_1$, $a_2$, $b$, $d$, $h$ 和 $R$。

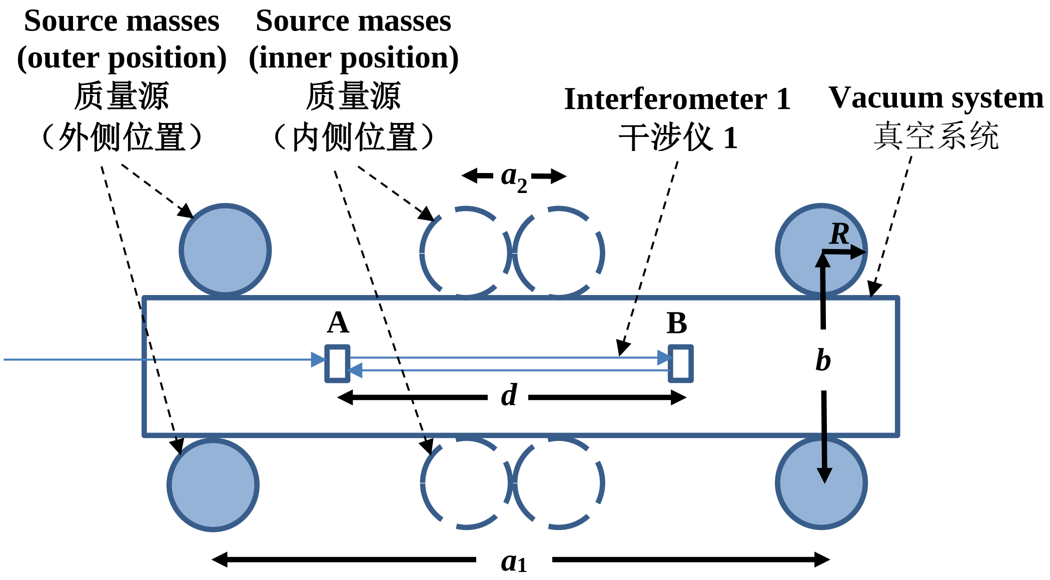


*Figure 2: Top view of the apparatus. 仪器的顶视图。*

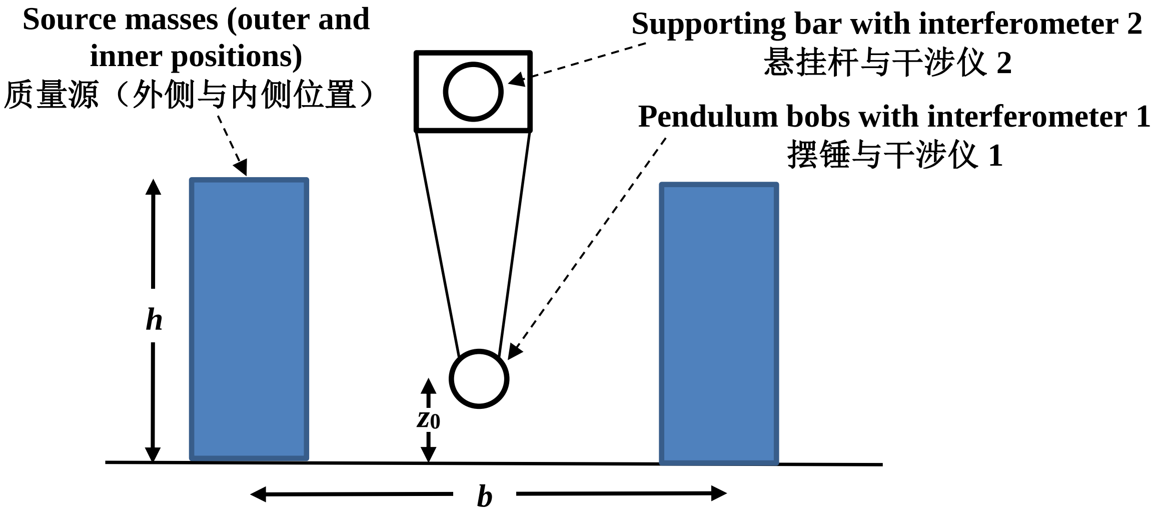


*Figure 3: The side view of the apparatus. 仪器的侧视图。*

It is very complicated to calculate the gravitational force of the 4 cylindrical source masses acting on pendulum bob A. Here we approximate each cylinder with uniform density, mass $M$, radius $R$ and height $h$ to be a thin wire with uniform density, mass $M$ and height $h$ passing through the axis of the cylinder.

计算四个圆柱形质量源对摆锤 A 的引力非常复杂。这里，我们将每个密度均匀、质量为$M$、半径为 $R$、高度为 $h$ 的圆柱近似为沿圆柱轴向放置的细线，密度均匀、质量为 $M$、高度为 $h$ 。

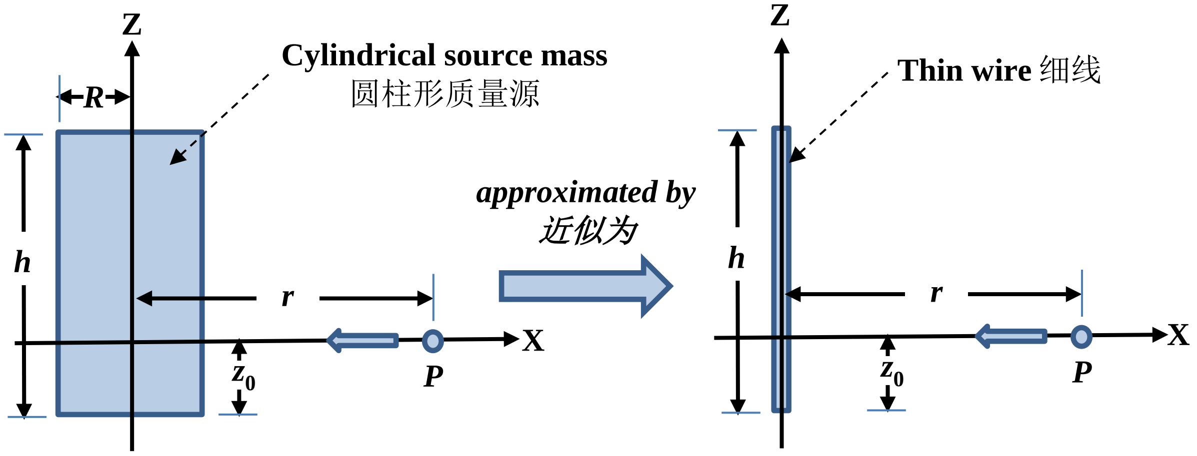


*Figure 4: The X-component of the gravitational field at point P due to a cylindrical source mass is now approximated by that due to a thin wire. 计算在 P 点引力场的 X 分量时，将圆柱形质量源近似为细线。*

You are provided the integral formula:
你可以使用以下积分公式：

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C,$$

| | | |
|---|---|---|
| A2 | Derive an expression of the X-component of the gravitational field $g_x$ at point $P$ due to the thin wire. Express your answer in terms of $G$, $M$, $h$, $z_0$, and $r$.<br>推导由细线产生的引力场在 $P$ 点的 X 分量 $g_x$。用 $G, M, h, z_0$ 和 $r$ 表示你的结果。 | 3 points<br>3 分 |

Consider an element of the plate at position $z$.
Distance from $P = \sqrt{r^2+z^2}$. [0.5]
Gravitational field at $P = \frac{G\rho\, dz}{r^2+z^2}$. [0.5]
X-component of the gravitational field $= \frac{G\rho\, dz}{r^2+z^2}\frac{r}{\sqrt{r^2+z^2}}$. [0.5]
Linear density: $\rho = \frac{M}{h}$.
Total X-component of the gravitational field

$$g_x = \frac{GM}{h}\int_{-z_0}^{h-z_0} dz \frac{r}{(r^2+z^2)^{\frac{3}{2}}} = \frac{GM}{h}\frac{rz}{r^2\sqrt{r^2+z^2}}\bigg|_{-z_0}^{h-z_0}$$ [integral expression 0.5, integration result 0.5]

$$= \frac{GM}{hr}\left(\frac{h-z_0}{\sqrt{r^2+(h-z_0)^2}} + \frac{z_0}{\sqrt{r^2+z_0^2}}\right).$$ [correct answer 0.5, no penalty for ± sign]

| | | |
|---|---|---|
| A3 | Calculate the gravitational field $g_x$ at point $P$ when the point is very near the thin wire.<br>当点 $P$ 非常接近细线时，计算引力场在 $P$ 点的 X 分量 $g_x$。 | 1 point<br>1 分 |

When the point is very near to the thin wire, $r$ approaches 0. Hence
$g_x = \frac{2GM}{hr}$. [noting that the two terms in the bracket approach 1: 0.5, correct answer 0.5]

| | | |
|---|---|---|
| A4 | Applying Gauss' law, verify the result in part A3. Write your steps in the answer sheet.<br>应用高斯定律，验证 A3 中的结论。在答题纸上写下过程。 | 1 point<br>1 分 |

Total flux of gravitational field:

$$\Phi = \left(\frac{GM}{r^2}\right)(4\pi r^2) = 4\pi GM. \qquad [0.5]$$

Applying Gauss' law to a cylindrical surface of height $\Delta z$ around the thin wire,

$$\Phi = 4\pi\frac{M}{h}\Delta z = g_x(2\pi r\Delta z).$$

$$g_x = \frac{2GM}{hr}. \qquad [0.5, \text{ no penalty for } \pm \text{ sign}]$$

You are provided the following parameters:
你可以使用以下参数：

| | | |
|---|---|---|
| $G = 6.67 \times 10^{-11}$ Nkg$^{-2}$m$^2$ | $M = 119.1$ kg | $a_1 = 0.568$ m |
| $a_2 = 0.166$ m | $b = 0.262$ m | $d = 0.34$ m |
| $h = 0.312$ m | $z_0 = 0.002$ m | $R = 0.083$ m |

| | | |
|---|---|---|
| A5 | Using the given parameters, and the thin wire approximation for the 4 cylindrical source masses, calculate the horizontal component of the gravitational field due to the 4 source masses at the position of pendulum bob A, when the source masses are located at the inner position.<br>使用上面给出的参数，以及用细线近似四个质量源。当质量源处于内侧位置时，计算摆锤 A 处四个质量源产生的引力场的水平分量。 | 3 points<br>3 分 |

For pendulum bob A, the 2 sources on the left and right in Fig. 3 are at distances

$$r_{left}^2 = \left(\frac{d}{2} - \frac{a_2}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.17 - 0.083)^2 + 0.131^2 = 0.02473 \text{ m}^2,$$

$$r_{right}^2 = \left(\frac{d}{2} + \frac{a_2}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.17 + 0.083)^2 + 0.131^2 = 0.08117 \text{ m}^2.$$

[know how to calculate: 0.5, correct substitution: 0.5]

The angles between the components of their gravitational fields and the axis of the interferometer are

$$\cos\theta_{left} = \frac{d - a_2}{\sqrt{(d - a_2)^2 + (b)^2}} = \frac{0.17 - 0.083}{\sqrt{(0.17 - 0.083)^2 + 0.131^2}} = 0.5532,$$

$$\cos\theta_{right} = \frac{d + a_2}{\sqrt{(d + a_2)^2 + (b)^2}} = \frac{0.17 + 0.083}{\sqrt{(0.17 + 0.083)^2 + 0.131^2}} = 0.8880.$$

[know how to calculate: 0.5, correct substitution: 0.5]

Hence the gravitational field along the interferometer axis due to the left source masses

$$g_{left} = (2)\frac{(6.67\times 10^{-11})(119.1)}{(0.312)\sqrt{0.02473}}\left(\frac{0.31}{\sqrt{0.02473+0.31^2}}+\frac{0.002}{\sqrt{0.02473+0.002^2}}\right)(0.5532)$$
$$= 1.6204\times 10^{-7}\ \mathrm{ms^{-2}}.$$
$$g_{right} = (2)\frac{(6.67\times 10^{-11})(119.1)}{(0.312)\sqrt{0.08117}}\left(\frac{0.31}{\sqrt{0.08117+0.31^2}}+\frac{0.002}{\sqrt{0.08117+0.002^2}}\right)(0.8880)$$
$$= 1.1798\times 10^{-7}\ \mathrm{ms^{-2}}.$$

Total gravitational field (in the rightward direction): $g_{total} = g_{left} + g_{right} = 2.8002\times 10^{-7}\ \mathrm{ms^{-2}}$.

[know how to calculate and sum the two: 0.5, correct substitution: 0.5]

| | | |
|---|---|---|
| A6 | Similar to part A5, calculate the horizontal component of the total gravitational field at the position of pendulum bob A, when the source masses are located at the outer position.<br>与 A5 部分类似，当质量源处于外侧位置时，计算摆锤 A 处总引力场的水平分量。 | 3 points<br>3 分 |

For pendulum bob A, the 2 sources on the left and right in Fig. 3 are at distances

$$r_{left}^2 = \left(\frac{a_1}{2}-\frac{d}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.284-0.17)^2 + 0.131^2 = 0.03016\ \mathrm{m}^2,$$
$$r_{right}^2 = \left(\frac{a_1}{2}+\frac{d}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = (0.284+0.17)^2 + 0.131^2 = 0.02233\ \mathrm{m}^2.$$

[know how to calculate: 0.5, correct substitution: 0.5]

The angles between the components of their gravitational fields and the axis of the interferometer are

$$\cos\theta_{left} = \frac{a_1 - d}{\sqrt{(a_1-d)^2+(b)^2}} = \frac{0.284-0.17}{\sqrt{(0.284-0.17)^2+0.131^2}} = 0.6565,$$
$$\cos\theta_{right} = \frac{a_1 + d}{\sqrt{(a_1+d)^2+(b)^2}} = \frac{0.284+0.17}{\sqrt{(0.284+0.17)^2+0.131^2}} = 0.9608.$$

[know how to calculate: 0.5, correct substitution: 0.5]

Hence the gravitational field along the interferometer axis due to the left source masses

$$g_{left} = (2)\frac{(6.67\times 10^{-11})(119.1)}{(0.312)\sqrt{0.03016}}\left(\frac{0.31}{\sqrt{0.03016+0.31^2}}+\frac{0.002}{\sqrt{0.03016+0.002^2}}\right)(0.6565)$$
$$= 1.7016\times 10^{-7}\ \mathrm{ms^{-2}}.$$
$$g_{right} = (2)\frac{(6.67\times 10^{-11})(119.1)}{(0.312)\sqrt{0.02233}}\left(\frac{0.31}{\sqrt{0.02233+0.31^2}}+\frac{0.002}{\sqrt{0.02233+0.002^2}}\right)(0.9608)$$
$$= 5.7237\times 10^{-7}\ \mathrm{ms^{-2}}.$$

Total gravitational field (in the leftward direction): $g_{total} = g_{left} - g_{right} = 1.1292\times 10^{-7}\ \mathrm{ms^{-2}}$.

[know how to calculate and subtract: 0.5, correct substitution: 0.5]

| | | |
|---|---|---|
| A7 | Calculate the change $\Delta g_x$ of the horizontal component of the gravitational field at the position of pendulum bob A, when the source masses are moved from the outer position to the inner position.<br>当质量源从外侧位置移到内侧位置，计算摆锤 A 处引力场水平分量的变化$\Delta g_x$。 | 1 point<br>1 分 |

Change in the horizontal component of the gravitational field
$= 2.8002 \times 10^{-7} + 1.1292 \times 10^{-7} = 3.9294 \times 10^{-7}\ \mathrm{ms}^{-2}$.
[know how to add the answers of A5 and A6: 0.5, correct answer: 0.5]

In this experiment, care has to be taken to monitor the uncertainties of measurements. Since the calculation is complicated, we will simply focus on the expression derived in Part A3. Consider the case that all mass measurements have an uncertainty of 0.6 parts in $10^5$, and all dimension measurements have an uncertainty of 1.4 parts in $10^5$.

本实验需要小心考虑实验误差。完整计算比较复杂，为了简化，我们使用 A3 部分中推导的表达式。假设所有质量测量的误差均为 $0.6 \times 10^{-5}$，所有长度测量的误差均为 $1.4 \times 10^{-5}$。

Remark: The uncertainty of a physical quantity $f(x_1, x_2, \ldots)$ is given by the standard deviation of $f$ divided by $f$, that is, $\sqrt{\sigma_f^2/f^2}$, where $x_1, x_2, \ldots$ are independent measurements and $\sigma_f^2$ is calculated from $\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$.

注：物理量 $f(x_1, x_2, \ldots)$的误差为 $f$ 的标准差除以 $f$ 本身，即$\sqrt{\sigma_f^2/f^2}$，其中 $x_1, x_2, \ldots$为独立测量的物理量，$\sigma_f^2$ 可以用 $\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots$ 计算。

| | | |
|---|---|---|
| A8 | Calculate the uncertainty of $g_x$ in Part A3.<br>计算 A3 部分中 $g_x$ 的误差。 | 2 points<br>2 分 |

$$\sigma_{g_x}^2 = \left(\frac{2G}{hr}\right)^2 \sigma_M^2 + \left(-\frac{2GM}{h^2 r}\right)^2 \sigma_h^2 + \left(-\frac{2GM}{hr^2}\right)^2 \sigma_r^2.$$

$$\frac{\sigma_{g_x}^2}{g_x^2} = \frac{\sigma_M^2}{M^2} + \frac{\sigma_h^2}{h^2} + \frac{\sigma_r^2}{r^2} = (0.6 \times 10^{-5})^2 + (1.4 \times 10^{-5})^2 + (1.4 \times 10^{-5})^2 = 4.28 \times 10^{-10}.$$

$$\frac{\sigma_{g_x}}{g_x} = \sqrt{4.28 \times 10^{-10}} = 2.1 \times 10^{-5}.$$

[correct formula 1, correct answer 1]

## Part B. Measurement of the Gravitational Field Change During the Experiment (15 points)测量实验中的引力场改变（15 分）

In Part A we estimated the gravitational field change when the source masses are moved from the outer to the inner position, assuming a certain value of the gravitational constant $G$. In this part we investigate how the gravitational field change can be measured experimentally, such that the gravitational constant $G$ can be calculated. We let:

$\Delta x$ = the horizontal displacement of the pendulum bob A when the source masses are moved from the outer to the inner position,
$\omega$ = the angular frequency of the natural oscillations of the pendulum bob,
$m$ = mass of the pendulum bob.

在 Part A 中，我们假设给定引力常数 $G$，估算了质量源从外侧移到内侧时引力场的改变。本部分中，我们将研究如何在实验中测量引力场的改变，以及如何因此测量引力常数 $G$。我们定义三个量：

$\Delta x$ = 当质量源从外侧移到内侧时，摆锤 A 在水平方向的偏移量，
$\omega$ = 单摆自然振动的角频率，
$m$ = 摆锤的质量。

| | | |
|---|---|---|
| B1 | Derive the expression of the change $\Delta g_x$ in the horizontal component of the gravitational field at the position of pendulum bob A, when the source masses are moved from the outer position to the inner position. Express the result in terms of the 3 variables above.<br>当质量源从外侧移到内侧，求摆锤 A 处引力场水平分量的改变 $\Delta g_x$。用以上三个量表示结果。 | 1 point<br>1 分 |

$$\Delta g_x = g\tan\theta \approx g\theta = g\frac{\Delta x}{l} = \omega^2 \Delta x.$$

[correct steps 0.5, correct answer 0.5]

The pendulum bobs are hung from the supporting bar at a distance $l$ = 0.738 m vertically below the bar. The gravitational acceleration is $g$ = 9.8 ms$^{-2}$.

摆锤从顶部悬挂下来，与悬挂杆的垂直距离为 $l$ = 0.738 m。引力加速度是 g = 9.8 ms$^{-2}$。

| | | |
|---|---|---|
| B2 | Using your result in part A7, calculate the change in the separation of the pendulum bobs.<br>利用 A7 部分的结果，计算摆锤位置的变化量。 | 1 point<br>1 分 |

$$\Delta g_x = g\frac{\Delta x}{l}$$

$$\Delta x = l\frac{\Delta g_x}{g} = (0.738)\left(\frac{3.9294\times 10^{-7}}{9.8}\right) = 29.5912\ \text{nm}.$$

[correct steps 0.5, correct answer 0.5]

There are corrections to the answer in B2 because besides the pendulum motion, there are other contributions to $\omega^2$ such as the flexing of the wires. The frequencies were found to be (0.589 8171 ± 0.000 0023) Hz for one bob and (0.589 7069 ± 0.000 0013) Hz for the other, where the number following the plus-minus sign is the standard derivation of the quantity.

B2 中的答案还没有加入校正，因为除了单摆运动之外，还有其它因素影响$\omega^2$，例如摆线的弹性。对一个摆锤，频率的测量值为 (0.589 8171 ± 0.000 0023) Hz，对另一个摆锤，频率的测量值为 (0.589 7069 ± 0.000 0013) Hz，其中正负号后面的数字为该量的标准差。

| | | |
|---|---|---|
| B3 | Calculate the mean and the uncertainty of the average value of the pendulum frequency.<br>计算单摆频率的平均值，以及单摆频率平均值的误差。 | 2 points<br>2 分 |

Mean:

$$\frac{1}{2}(0.589\,8171 + 0.589\,7069) = 0.589\,7620\text{ Hz}. \qquad [1]$$

Uncertainty:

$$\frac{\sqrt{\frac{1}{4}(0.000\,0023)^2 + \frac{1}{4}(0.000\,0013)^2}}{0.589\,7620} = 2.2 \times 10^{-6}. \qquad [1]$$

| | | |
|---|---|---|
| B4 | If the error of time measurement is $10^{-5}$ s, determine the number of periods to be measured such that the uncertainty of the measured period is $10^{-7}$.<br>假设时间测量的绝对误差为$10^{-5}$秒，为了达到误差为 $10^{-7}$ 的周期测量精度，求需要测量的周期数量。 | 1 point<br>1 分 |

Period:

$$T = \frac{1}{0.589\,7620} = 1.6956\text{ s}. \qquad [0.5]$$

Let $n$ be the number of periods. Then

$$\frac{10^{-5}}{nT} = 10^{-7}.$$

$$n = \frac{10^{-5}}{10^{-7}T} = 59. \qquad [0.5]$$

In practice, the pendulum was swung over several hours of measurement to check whether the pendulums are stable, and repeated over the course of several months.

The horizontal displacement of the pendulum bobs can be measured with high precision using the laser interferometer (commonly known as Fabry-Pérot interferometer). The cavity of the interferometer is formed by two highly reflective mirrors separated at a distance $d$ = 0.34 m so that the laser team traveling between them forms a standing wave. The laser wavelength is 633 nm. When the spectrum is closely examined, one finds that it consists of a sequence of peaks as shown in Fig. 5.

摆锤的水平位移可以通过激光干涉仪（法布里-珀罗干涉仪）精确测量。干涉仪腔体的两端分别放置了一块反射系数很高的镜片。两镜片相距$d$ = 0.34 m。激光在两镜片之间形成驻波。激光的波长为633 nm。如图 5 所示，干涉仪的光谱由一系列峰组成。

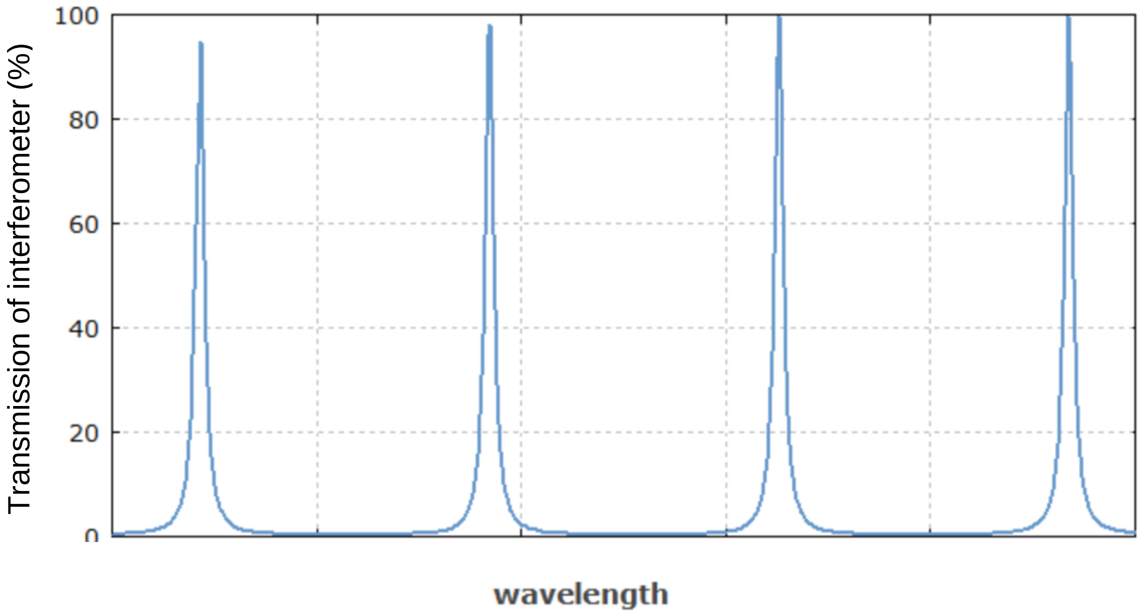


Figure 5: A sketch of the close-up of the spectrum of the interferometer. 干涉仪中光谱的细致结构示意图。

| | | |
|---|---|---|
| B5 | Calculate the frequency separation of the neighboring peaks in the spectrum. Then calculate the wavelength separation of the neighboring peaks in the spectrum.<br>计算光谱中相邻两峰之间的频率差，并计算光谱中相邻两峰之间的波长差。 | 2 points<br>2 分 |

The peaks correspond to different modes of standing waves. Neighboring peaks correspond to standing waves differing in their traveling time back and forth the interferometer by one period. Hence the frequency separation of the neighboring peaks is given by

$$\Delta f = \frac{c}{2L} = \frac{3\times10^8}{0.68} = 441\ \text{MHz}. \qquad [1]$$

Wavelength separation:

$$\Delta\lambda = \frac{\lambda}{f}\Delta f = \frac{\lambda^2}{c}\Delta f = \frac{\left(633\times10^{-9}\right)^2}{3\times10^8}\left(4.41\times10^8\right) = 5.89\times10^{-4}\ \text{nm}. \qquad [1]$$

The width of the fringes in Fig. 5 is about 100 kHz.
图 5 中，条纹的宽度约为 100kHZ。

| | | |
|---|---|---|
| B6 | Assuming that the major mechanism of power loss of the standing wave is the transmission through the mirrors, estimate the fraction of power loss from the interferometer per transmission.<br>假设驻波中主要的能量损失来自穿过镜面的透射。估算每次透射中能量损失占总能量的比例。 | 2 points<br>2 分 |

When there is no input power, the time taken by the phase angle of the standing wave mode to spread out over a range of $2\pi$ completely is

$\frac{1}{(2\pi)(100\times10^3)}$. [1]

Time for the standing wave to travel between successive transmission is

$\frac{L}{c}=\frac{0.34}{3\times10^8}$.

Hence the fraction of power loss per transmission is

$\left(\frac{0.34}{3\times10^8}\right)(2\pi)(100\times10^3)=7\times10^{-4}$. [1]

Remark: Answers without the factor of $2\pi$ are acceptable. In that case the answer will be $1\times10^{-4}$.

| | | |
|---|---|---|
| B7 | Suppose the length of the interferometer changes by 1 nm. Calculate the corresponding change in the cavity frequency.<br>设干涉仪长度改变 1 nm，求腔体中频率的相应改变。 | 2 points<br>2 分 |

The $n$th harmonic of the cavity frequency is given by

$$f=\frac{nc}{2L}.$$

Hence the change in the cavity frequency is given by

$$\delta f=-f\frac{\delta L}{L}=-\left(\frac{3\times10^8}{633\times10^{-9}}\right)\left(\frac{1\times10^{-9}}{0.34}\right)=-1.3939\text{ MHz}.$$

[correct equation of $\delta f$: 1, correct answer 1]

When the source masses are moved from the outer to the inner position, the beat frequency of the cavity frequencies of the two interferometers changes by 125 MHz.
当质量源从外侧移到内侧时，两个干涉仪腔体中频率的拍频改变了 125 MHz。

| | | |
|---|---|---|
| B8 | Calculate the change in the separation of the pendulum bobs. (Remark: This result will be different from that in B2 due to the approximations made in Part A.)<br>计算两个摆锤间距离的改变。（注：由于 Part A 中做的近似，这个结果将与 B2 中的不同。） | 1 point<br>1 分 |

Change in the separation: $\Delta x=\frac{125\times10^6}{1.3939\times10^6}=89.675$ nm. [1]

References:

[1] Harold V. Parks and James E. Faller. Simple Pendulum Determination of the Gravitational Constant. *Physical Review Letters* **105**, 110801 (2010).

[2] Harold V. Parks and James E. Faller. A simple pendulum laser interferometer for determining the gravitational constant. *Philosophical Transactions of the Royal Society* **A372**, 20140024 (2014).

## Problem 2: Physics in Various Dimensions (33 points) 不同维度的物理 (33 分)

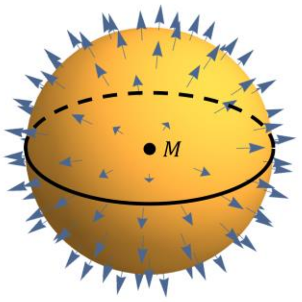
“What had she experienced? She had seen how a cruel attacker could lower the dimensions of space by one and destroy a solar system. What are dimensions?” -- Cixin LIU, *Death’s End* (Translation: Ken LIU)

“她经历过什么？她刚刚看到，为了毁灭一个恒星系。残忍的攻击者把那里的空间维度降低了一维。空间维度，空间维度是什么？”—— 刘慈欣《三体・死神永生》

In this problem, we will explore 4-dimensional space, attack from 4-dimensional to 3-dimensional spaces, and the motion of celestial bodies in two dimensions. (Note: In this problem, whenever we mention the number of dimensions, we mean spatial dimensions and did not count in the time dimension. For example, when we mention 3-dimensional space, we mean the spacetime with 3 space dimensions and 1 time dimension). The setup of the problem is as follows:

本题中，我们将讨论四维空间、从四维到三维的维数攻击，以及二维空间中的天体运动问题。（注意：本题提到空间维度的数量时，并没有把时间维计算在内。例如，当我们提到三维空间时，指的是三维空间加上一维时间组成的时空。）问题设定如下：

- Let the mass of the star be $M$， the mass of the planet be $m \ll M$， So in the star-planet problems, the star can be considered at rest. The change of star position due to planet motion can be neglected.
- In $n$-dimensional space, Newtonian gravity can still be derived from Gauss flux law: For any (n-1)-dimensional closed surface enclosing the point particle $M$, the gravitational acceleration flux through the surface is $-A_{n-1}GM$, where $A_{n-1}$ is the area of a $(n-1)$-dimensional unit sphere (understood as generalized area, for example, for $n = 2$, $A_1 = 2\pi$ is length, for $n = 3, A_2 = 4\pi$ is area, and for $n = 4$, $A_3 = 2\pi^2$ is volume). The meaning of flux is: for a small area element, the flux is the dot product of the gravitational acceleration vector and the area vector. For the case when the gravitational acceleration is normal to the surface, flux is the magnitude of gravitational acceleration times this area. For example, the figure below illustrates how to derive Newtonian gravity from Gauss flux law for the case of 3 spatial dimensions.



Newtonian gravity from
Gauss flux law in 3 dimensions:

(flux through spherical surface)
$= (F/m) \times 4\pi r^2 = -A_2 GM$

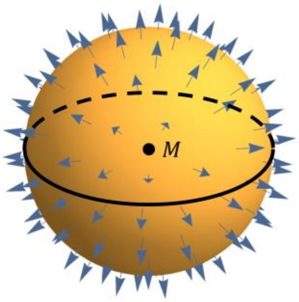
$\Rightarrow F = -GMm/r^2$

(Note: the Gauss flux law does not apply for general relativity.)

- Newton’s three laws of motion still holds. Momentum conservation and angular momentum conservation laws still holds. The relativity of motion still holds.
- We approximate the stars and planets as particles, with negligible radius.

- 设恒星质量为 $M$，行星质量 $m \ll M$，故在行星围绕恒星运动问题中，恒星可以视为静止，行星运动对恒星位置的影响可以忽略。

- 在 $n$ 维空间，牛顿引力可以由高斯引力通量定律推导出来，即对于包括了质点 $M$ 的任何 $n-1$ 维闭合曲面，穿出此闭合曲面的引力加速度通量等于 $-A_{n-1}GM$，其中 $A_{n-1}$ 为 $(n-1)$ 维单位球面的面积（理解为广义的面积，例如在 $n=2$ 情况下 $A_1=2\pi$ 为长度，$n=3$ 情况下 $A_2=4\pi$ 为面积，$n=4$ 情况下$A_3=2\pi^2$为体积）。通量的意思是：对一个小面积元，通量是引力加速度矢量点乘有向面积元。对引力加速度垂直于面积元的情况，通量为引力加速度的大小乘以面积元的面积。例如，下图中，对于三维空间，我们用高斯引理通量定律推导了牛顿引力定律。



三维空间中用高斯引力通量
定律推导牛顿引力定律

(穿过球面的引力加速度通量)
$= (F/m) \times 4\pi r^2 = -A_2GM$

$\Rightarrow F = -GMm/r^2$

（注意：高斯通量定律并不适用于广义相对论。）

- 牛顿三定律成立。动量守恒、角动量守恒定律成立。运动的相对性成立。
- 所有星球视为质点，其半径足够小，可以忽略。

**Part A. High-Dimensional World (13 points) 高维的世界(13 分)**

In the whole part A, we consider Newtonian gravity in $n$-dimensional $(n \geq 4)$ space. Considering that the gravitational acceleration vector is parallel to the position vector, the orbit of the planet is within the 2-dimensional plane determined by the position and velocity vectors of the planet.

在整个 Part A 中，我们考虑 $n$ $(n \geq 4)$ 维空间中的牛顿引力。由于引力加速度与位置矢量共线，行星围绕恒星的运动轨迹处于行星的位置矢量与速度矢量所决定的二维平面内。

| | | |
|---|---|---|
| **A1** | Applying the Gauss flux law to a $(n-1)$-dimensional sphere, one can derive the Newtonian law of gravity in $n$ space dimensions. Due to attraction from the star, the gravity force exerted on the planet is $F = -GMmr^{\alpha}$ (the direction of the force points to the star); the gravitational potential energy is $V(r) = GMm\, r^{\beta}/\beta$, where $r$ is the distance between the star and the planet. Express $\alpha, \beta$ in terms of $n$.<br>将高斯引力通量定律应用于 $(n-1)$ 维球面，可以推导 $n$ 维空间中的牛顿引力定律。由于受到恒星吸引，行星受到的引力为 $F = -GMmr^{\alpha}$ （力的方向指向恒星），引力势能为 $V(r) = GMm\, r^{\beta}/\beta$，其中 $r$ 为行星与恒星的距离，求 $\alpha, \beta$（用 $n$ 表示）。 | **2 point**<br>**2 分** |

Solution:

Using Gauss law, the area of a $(n-1)$ -dimensional sphere is $A_{n-1}r^{n-1}$. Thus, $F=-GMm/r^{n-1}$, i.e. $\alpha = 1-n$. (1p)

Either integrate $F=-\frac{GMm}{r^{n-1}}$, or from dimensional analysis, we get $\beta = 2-n$, i.e. $V=-\frac{GMm}{(n-2)\,r^{n-2}}$. (1p)

| | | |
|---|---|---|
| **A2** | The Kepler's second law in $n$ dimensions is: $L=mr^{\lambda}\dot{\varphi}$ is a constant (where $\varphi$ is the angle between the planet-star plane and the $x$-axis of the motion plane, $\dot{\varphi}\equiv d\varphi/dt$). Find $\lambda$.<br>$n$ 维的开普勒第二定律为： $L=mr^{\lambda}\dot{\varphi}$ 为常数（$\varphi$ 为行星-恒星连线与其运动平面的 $x$ 轴之间的夹角， $\dot{\varphi}\equiv d\varphi/dt$）。求 $\lambda$。 | **1 point**<br>**1 分** |

Solution:

$\lambda = 2$, same as the 3-dimensional case, since the motion is restricted in 2-dimensional plane.

| | | |
|---|---|---|
| **A3** | Calculate the speed of the planet along the radius direction $\dot{r}\equiv dr/dt$. Express your result using $n,r,G,M,m,E,L$, where $E$ is the energy of the planet (including kinetic energy and gravitational potential energy).<br>求行星的径向速度 $\dot{r}\equiv dr/dt$，用 $n,r,G,M,m,E,L$ 表示，其中 $E$ 为行星的能量（包括动能和引力势能）。 | **3 points**<br>**3 分** |

Solution:

Kinetic energy $E_K$ has two parts (perpendicular to each-other): along r direction and along $\phi$ direction. Thus, $E_K=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\,\dot{\phi}^2$. (1p)

Using Kepler's law to replace $\dot{\phi}$ by $L$: $E_K=\frac{1}{2}m\dot{r}^2+\frac{L^2}{2mr^2}$. (0.5p)

Energy conservation: $E=E_K+V$ (0.5p)

Solve $\dot{r}$ from above: $\dot{r}=\pm\sqrt{\frac{2E}{m}+\frac{2GM}{(n-2)r^{n-2}}-\frac{L^2}{m^2r^2}}$ . (1p) (#)

(If only plus sign is given in above $\pm$, deduct 0.5p.)

| | | |
|---|---|---|
| **A4** | Give the conditions for the planet to form a circular orbit (give algebraic equations using $n,r,G,M,m,E,L$, no need to solve these equations). | **2 points**<br>**2 分** |

| | | |
|---|---|---|
| | 给出行星沿圆轨道运动的所有条件（给出关于 $n, r, G, M, m, E, L$ 的代数方程组即可，不必解方程）。 | |

Solution: The two necessary conditions for circular orbit are:

(1) $\dot{r} = 0$ (0.5p), i.e. $\frac{2E}{m} + \frac{2GM}{(n-2)r^{n-2}} - \frac{L^2}{m^2 r^2} = 0$ (0.5p) (*)

(2) $\ddot{r} = 0$ (or equivalently gravitational force balance centripetal acceleration) (0.5p),

i.e. $\frac{GM}{r^{n-1}} - \frac{L^2}{m^2 r^3} = 0$ (0.5p) (**)

| | | |
|---|---|---|
| **A5** | For $n = 4$, when the values of $r, G, M, m, E, L$ are such that a circular orbit is possible, and $r$ is varied while other parameters are fixed, can the planet<br>(1) form elliptical orbits?<br>(2) move from finite $r$ to $r \to \infty$?<br>(3) move from finite $r$ to $r \to 0$?<br>当 $n = 4$，并且 $r, G, M, m, E, L$ 的取值使得行星有可能形成圆轨道时，固定其它参数而 $r$ 取不同的值时，行星是否有可能<br>(1) 形成椭圆轨道<br>(2) 从有限的 $r$ 运动到 $r \to \infty$<br>(3) 从有限的 $r$ 运动到 $r \to 0$ | **1.5 points**<br>**1.5 分** |

Solution:

For $n = 4$: equations (*) and (**) implies $E = 0$. Then a solution for one $r$ is a solution for all values of $r$. Thus, for any $r$, we have circular orbit $\dot{r} = 0$. Thus, all (1), (2), (3) are impossible. (0.5p each)

| | | |
|---|---|---|
| **A6** | For $n > 4$, when the values of $n, r, G, M, m, E, L$ are such that circular orbit is possible, and $r$ is varied while other parameters are fixed, can the planet<br>(1) form elliptical orbits?<br>(2) move from finite $r$ to $r \to \infty$?<br>(3) move from finite $r$ to $r \to 0$?<br>For the possible cases in the above questions, please state the possible range of $r$ given $n, r, G, M, m, E, L$ are fixed. (If the limit of the range is one of the roots of an algebraic equation, please specify which root without the need to solve the equation explicitly.)<br>当 $n > 4$，并且 $n, r, G, M, m, E, L$ 的取值使得行星有可能形成圆轨道时，固定其它参数而 $r$ 取不同的值时，行星是否有可能 | **3.5 points**<br>**3.5 分** |

| | | |
|---|---|---|
| | (1) 形成椭圆轨道<br>(2) 从有限的 $r$ 运动到 $r \to \infty$<br>(3) 从有限的 $r$ 运动到 $r \to 0$<br>对于上面回答为可能的情况，在 $n, r, G, M, m, E, L$ 固定的情况下，请指出 $r$ 的取值范围。（如果 $r$ 取值范围的边界为代数方程的一个根，请指出哪个根，但不必解这个方程。） | |

Solution:

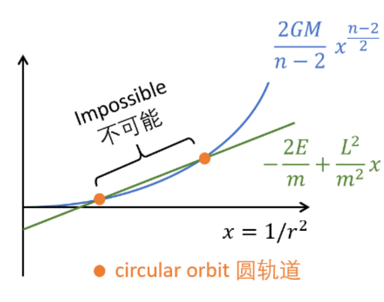
For $n > 4$: equations (*) and (**) implies $E > 0$. (0.5 p)

For (#) to make sense (positivity under the square root), we need (0.5p)

$$\frac{2E}{m} + \frac{2GM}{(n-2)r^{n-2}} - \frac{L^2}{m^2 r^2} \geq 0$$

The solutions of this inequality can be studied by letting $x = \frac{1}{r^2}$, and plot & compare the functions $\frac{2GM}{n-2} x^{(n-2)/2}$ and $\frac{L^2}{m^2} x - \frac{2E}{m}$ (0.5p).

The result is:



There are two positive $r$ solutions of $\frac{2E}{m} + \frac{2GM}{(n-2)r^{n-2}} - \frac{L^2}{m^2 r^2} = 0$ (0.5p). Let us denote them by $r_1, r_2$ ($r_1 \leq r_2$). Then

(1) is impossible. (0.5p)

(2) Condition for (2) is $r < r_1$. (0.5p)

(3) Condition for (3) is $r > r_2$. (0.5p)

**Part B. Dimensional attacks in Newtonian gravity (5 points) 牛顿引力下的维度打击 (5 分)**

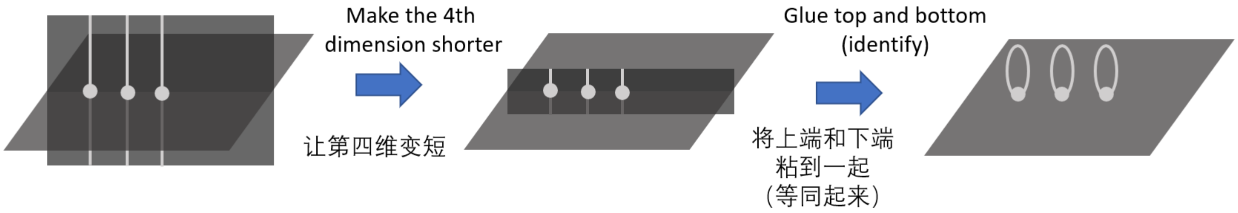
Suppose aliens living in 4 spatial dimensions perform dimensional attack on enemies also living in 4 spatial dimensions. The way of dimensional reduction is to reduce one space dimension into a small circle with circumference length $C$. In this way, in fact, there are still 4 spatial dimensions. But for small $C$ values, seeing from far away, one cannot see the dimension of the small circle. As a result, seeing from far away, the enemies appear to be living in 3 spatial dimensions. We call such spatial regions under dimensional attack having “effectively” three spatial dimensions.

假设四维空间中的生物向同样生活在四维的敌方实施维度打击。降维的方法是，将一个空间维度缩小成周长为 $C$ 的小圆环。这样，其实空间还有四维。但是当 $C$ 很小时，从远处看来，看不到这个圆

环代表的维度。这样，从远处看来，敌方就好像生活在三维空间一样。我们称这时遭受维度打击的空间区域“有效”维度为三维。

As illustrated in the below figure, suppose the vertical dimension is the dimension under dimensional attack. The horizontal plane denotes the remaining three-dimensional space. Before the dimensional attack, the four-dimensional space can be considered as, in a three-dimensional space, on every point there is one straight line indicating the fourth dimension. After the dimensional attack, on every point there is a small circle with circumference $C$ indicating the fourth dimension. The circle indicates that the top and bottom endpoints of an interval are glued.

如下图所示，假设垂直方向的维度为实施维度打击的维度，水平的平面代表剩余的三维空间。维数打击前，四维空间可以看成：三维空间的每个点上都有一条直线代表第四维。维数打击后，三维空间的每个点上都有一个周长为 $C$ 的小圆环代表第四维。圆环的含义为将一条线段的上下两个端点粘起来。



Assume that during the dimensional attack (assume the duration is short enough), apart from the sudden change of the law of gravity, a system under dimensional attack does not have additional forces exerting on it. The momentum in the 4-dimensional point of view does not change. The law of gravity can be understood in two ways: the effectively three-dimensional gravity and the more fundamental four-dimensional gravity. The values of a gravitational force computed using these two methods (using Gauss flux law) agree with each other.

假设在维度打击过程中（假设此过程持续时间足够短），除了引力定律的形式忽然改变外，被打击的系统不受额外的力作用，四维观点下的动量不变。引力定律受维度打击的影响体现为:维数打击后可以从两种观点理解引力，有效的三维引力和更基本的四维引力，通过高斯定律用这两种观点算出的引力大小相等。

| | | |
|---|---|---|
| **B1** | Consider a point particle at rest, which exerts Newtonian gravity upon other objects (whose distance is much greater than $C$). Find the relation between the four-dimensional Newtonian gravitational constant $G_4$ and the effective three-dimensional Newtonian gravitational constant $G_3$ (in terms of $G_4 =$ (function of $G_3$ and other parameters)).<br>考虑一个静止质点对其它物体（距离远远大于 $C$ ）产生的牛顿引力。求四维牛顿引力常数 $G_4$ 和三维有效牛顿引力常数 $G_3$ 之间的关系（用 $G_4 =$（$G_3$及其它参数的函数） 表示）。 | **3 points**<br>**3 分** |

Solution:

In 4D, note that the new geometry is a cylinder. Now that $r \gg C$, the gravitational flux is approximately perpendicular to the circle (with perimeter C). The corresponding Gauss's Law is $4\pi r^2 C\ F = -2\pi^2 G_4 Mm$. (1p)

Compared with the 3D gravitational force is $F = -G_3 Mm/r^2$ (note: this 1p is given to realizing 4D force = 3D force, not knowing the 3D Newtonian gravity formula), we get $G_4 = \frac{2C}{\pi} G_3$. (1p)

[If the student guess $G_4 \propto CG_3$ with wrong proportional constant without reasons, give 1p.]

| | | |
|---|---|---|
| **B2** | Suppose the relativistic mass-energy relation $E^2 = \lvert\mathbf{p}\rvert^2 c^2 + m^2 c^4$ applies both for three and four dimensions. Here $c$ denotes the speed of light. Suppose before the dimensional attack, a point particle with mass $m$ has momentum $\mathbf{p} = (p_1, p_2, p_3, p_4)$. After dimensional attack, the 4th dimension (corresponding to the subscript 4 above) becomes a small circle. Calculate the three-dimensional effective mass of the point particle after the dimensional attack.<br>假设相对论质能关系 $E^2 = \lvert\mathbf{p}\rvert^2 c^2 + m^2 c^4$ 在三维和四维都成立，其中 $c$ 为光速。设维度打击之前，一个质量为 $m$ 的质点的动量为 $\mathbf{p} = (p_1, p_2, p_3, p_4)$，维度打击后，第四个空间维度（对应上述角标 4）变成小圆环，求质点在维度打击后的三维空间中的有效质量。 | **1 point**<br>**1 分** |

Solution: $m_{\text{eff}} = \sqrt{m^2 + \frac{p_4^2}{c^2}}$. (1p)

| | | |
|---|---|---|
| **B3** | Suppose a planet (with mass $m$) is moving along circular orbit in 4 space dimensions, with the star (with mass $M$) at the center. The distance between the planet and the star is $r$. Now dimensional attack this system along a direction perpendicular to the plane of planet motion. Calculate the energy of the planet after the dimensional attack (gravitational potential energy plus kinetic energy in the three-dimensional effective point of view).<br>设质量为 $m$ 的行星在四维空间中以圆轨道绕质量为 $M$ 的恒星运动，与恒星距离为 $r$。现对此星系沿垂直行星运动平面方向进行维度打击。求维度打击后，三维空间中行星的总能量（三维有效观点下的引力势能加动能）。 | **1 point**<br>**1 分** |

Solution:

Since gravity is much stronger in 3D, the three-dimensional energy is dominated by its potential energy $-\frac{G_3 Mm}{r}$ (1p)

It is also correct if more details are given (still 1p):

Before the dimensional attack, in 4D:

Spherical orbit: $\frac{L^2}{m} = G_4 Mm$.

The dimensional attack relates $G_4 = \frac{2C}{\pi}\, G_3$. Thus, $\frac{L^2}{m} = \frac{2CG_3Mm}{\pi}$.

The 3-dimensional energy is $E = -\frac{G_3Mm}{r} + \frac{CG_3Mm}{\pi\, r^2}$.

Remark: However, considering that $G_3$ receives $O(C/r)$ corrections, this form is not much more precise than $E \sim -\frac{G_3Mm}{r}$.

**Part C. The Two-Dimensional Newtonian World (4 points) 二维牛顿世界 (4 分)**

We further reduce spatial dimensions and consider Newtonian gravity in two dimensions ($n = 2$).

我们进一步减少空间维数，考虑二维（$n = 2$）空间中的牛顿引力。

| | | |
|---|---|---|
| **C1** | Provide formulas for the Newtonian gravitational force law and the corresponding gravitational potential energy in two dimensional.<br>求二维的牛顿万有引力公式及引力势能公式。 | **1 point**<br>**1 分** |

Solution:

$F = -\frac{GMm}{r}$ (0.5p),

$V = GMm \log r/r_0$ or $V = GMm \log r$ (0.5p)

| | | |
|---|---|---|
| **C2** | Provide relations for planets to move along circular orbits around the star (give algebraic equations in terms of $r, G, M, m, E, L$. You don’t need to solve the equation).<br>求行星绕恒星沿圆轨道运动的所有条件（给出关于 $r, G, M, m, E, L$ 的代数方程组即可，不必解方程）。 | **1 point**<br>**1 分** |

Solution:

$\frac{2E}{m} - 2GM\,(\log r + \text{const}) - \frac{L^2}{m^2r^2} = 0$ (0.5p)

(Note, here a constant is introduced to make the solution more general. The students don’t have to include the constant in their answer to get this 0.5p)

$\frac{GMm}{r} = \frac{L^2}{m^2r^2}$ (0.5p)

| | | |
|---|---|---|
| **C3** | When the values of $r, G, M, m, E, L$ are such that circular orbit is possible, and $r$ is varied while other parameters are fixed, can the planet<br>(1) move in a bounded range $r_1 \leq r \leq r_2$, where $0 < r_1 < r_2 < \infty$<br>(2) move from finite $r$ to $r \to \infty$?<br>(3) move from finite $r$ to $r \to 0$?<br>Just answer possible or impossible for the above three questions (may need to discuss different cases for different parameter choices).<br>当 $n, r, G, M, m, E, L$ 的取值使得行星有可能形成圆轨道时，固定其它参数而 $r$ 取不同的值时，行星是否有可能<br>(1) 始终在有限的 $r_1 \leq r \leq r_2$ 区间内运动，其中 $0 < r_1 < r_2 < \infty$<br>(2) 从有限的 $r$ 运动到 $r \to \infty$<br>(3) 从有限的 $r$ 运动到 $r \to 0$<br>对以上三问，分别回答能或不能即可（有可能要根据参数取值分情况讨论）。 | **2 point**<br>**2 分** |

Solution:

(1) Possible. Because both (2) and (3) are impossible. (1) must be possible. (Note: however, the orbit does not close to an ellipse, which needs a longer proof and is beyond the scope of this question. For those interested, one can search for Laplace-Runge-Lenz vector.) (0.5p)

However, there is one exception: if the two solutions of $\frac{2E}{m} - 2GM\,(\log r + \text{const}) - \frac{L^2}{m^2r^2} = 0$ coincide, then $r_1 = r_2$ , which does not leave room for the condition $0 < r_1 < r_2 < \infty$ (0.5p)

(2) The gravitational potential is infinitely deep. Thus, for any finite $E$, (2) is impossible. (0.5p)

(3) Impossible. Existence of spherical orbit → $L \neq 0$. Thus near $r \to 0, \dot{r}$ is imaginary, no physical meaning. Alternative but much more complicated explanation: repeat analysis similar to A4-A6, you will get the same conclusion that you cannot reach $r \to 0$. (0.5p)

## Part D. The Two-Dimensional Einstein World (11 points) 二维爱因斯坦世界 (11 分)

Interestingly, in the two-dimensional case, the “gravitational” law in Einstein’s general relativity is even simpler than Newtonian gravity. In general relativity, there is no gravitational force at all between massive point particles (the space outside the particles is not curved either). Instead, the only gravitational effect of a point mass is that the space around it becomes conical (like a cone).

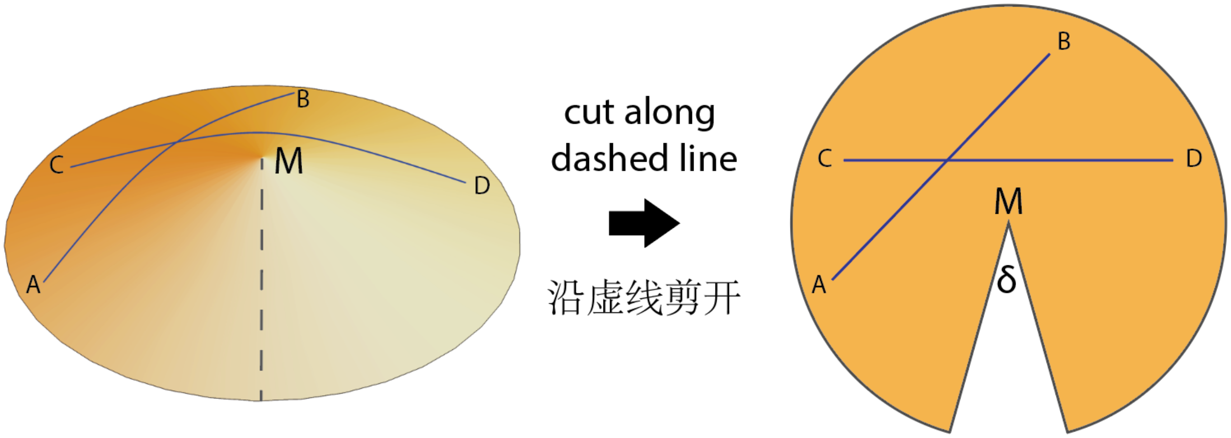
有趣的是，在二维空间情况下，爱因斯坦广义相对论中的“引力”规律比牛顿引力还要简单。广义相对论中，二维空间的质点之间根本没有相互吸引力（质点外的空间也没有弯曲）。而质量带来的效应为，一个质点周围的空间为圆锥形。

In the conical two-dimensional world, free particles and light will move along straight lines. Here straight lines are understood in the following way: if we cut the cone along a ray starting from the top vertex (not intersecting with the motion trajectory), and lay it on the plane as a circular sector, the motion trajectory is a straight line on the sector. For example, the lines AB and CD in the figure below.

在圆锥形的二维世界中，自由粒子和光线将沿直线运动。这里的直线理解为，将圆锥沿着任意一条从顶点出发、与运动轨迹不相交的射线剪开，并在平面上摊平成为扇形后，运动轨迹在扇形上呈直线。例如下图中的直线 AB，CD。

When we cut the cone into a sector, there is a deficit angle (the angle that a sector lacks compared to a disk), denoted by $\delta$ as illustrated in the figure below. This deficit angle is proportional to the mass $M$ of the point particle. Here we assume $\delta < \pi$.

把圆锥剪开成扇形时，扇形与圆盘相比所缺的角度（下图中的 $\delta$）称为圆锥的缺陷角，与质点的质量 $M$ 成正比。本题假设 $\delta < \pi$。



| | | |
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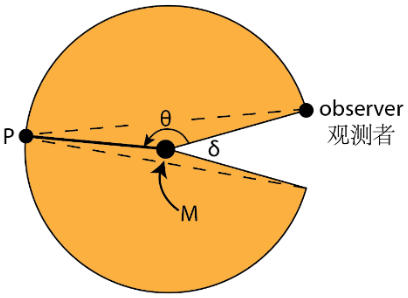
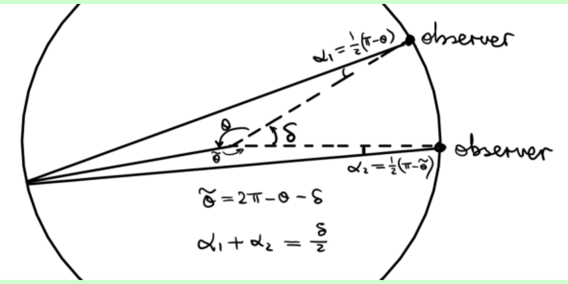
| **D1** | Point P is moving along a circle surrounding point M (M is the center of the sector in the figure below). The angle between the observer-M line and the MP line (viewed anti-clockwise) is $\theta$. Depending on different values of $\theta$, sometimes the observer finds a single image of P and sometimes finds double images of P. For example, in the figure below, the observer can observe double images along the dashed lines. Find the condition for the observer to observe double images, and the angle between these two images (the angle between the two dashed lines in the figure).<br>点 P 围绕质点 M 做圆周运动(M 为图中扇形中心)。观测者在同一个圆周上观测 P 点的运动。观测者与 M 连线与 MP 连线（沿逆时针方向看去）的夹角为 $\theta$。随 $\theta$ 取值不同，观测者有时看到 P 的单像，有时看到双像。例如右图，观<br><br> | **2 points**<br>**2 分** |

| | | |
|---|---|---|
| | 测者可以沿图中虚线看到 P 的双像。求观测者能看到 P 双像的条件，以及双像间的夹角（即图中两虚线间的夹角）。 | |

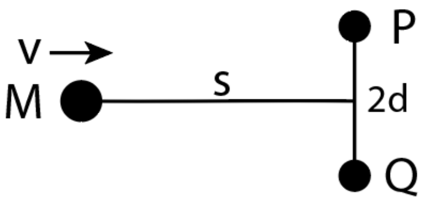
Solution:

(1) When $\pi - \delta < \theta < \pi$ double images, otherwise single image (1p)

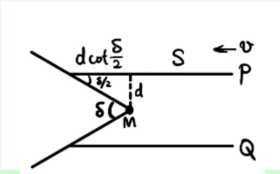


(2) Angle between the two images: the angle is $\delta/2$ (1p)

(A simpler observation is to use the inscribed angle theorem and get $\delta/2$ directly for this 1p.)

| | | |
|---|---|---|
| **D2** | At $t = 0$, a point particle with mass $M$ (with corresponding deficit angle $\delta$) moves towards observers P and Q. The direction of motion is perpendicular to the PQ interval, and the speed is $v$ (much smaller than the speed of light). The mass of P and Q are negligible. At the initial time $t = 0$, P and Q are at rest, distances MP = MQ, PQ = 2d, the distance between M and the PQ interval is $s$. Calculate the time $t_m$ when P and Q meet.<br>质量为 $M$（对应角度缺陷 $\delta$）的质点向两个质量可忽略的观测者 P、Q 运动，运动方向垂直于 PQ 连线，速率为 $v$（远小于光速）。在初始时刻 $t = 0$ 时，P、Q 静止，线段 MP = MQ，PQ = 2d，M 与 PQ 连线距离为 $s$。求 P、Q 两观测者相遇的时间 $t_m$。<br><br> | **2 points**<br>**2 分** |

Solution:



Though not necessary, it’s convenient to work in the frame where M is static, and P&Q are moving. Note that the deficit angle can be drawn facing any angle (in the above examples we have drawn it facing forward, but it can also be drawn facing backward). This is not necessary either, though it will simplify the discussion. This setup or an equivalent figure deserves 1p.

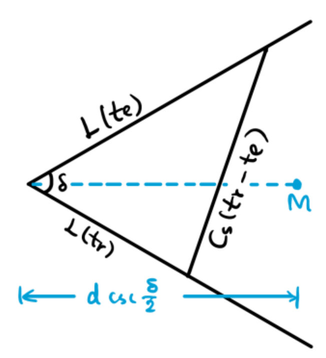
The meeting time is thus $t_m = \left[s + d\cot\left(\frac{\delta}{2}\right)\right]/v$ (1p).

| | | |
|---|---|---|
| **D3** | Consider the same setup as Problem D2. The observer P (starting from early enough time) continuously emits sound wave towards all directions. The source of sound has vibration frequency $f$. The speed of sound is $c_s$ satisfying $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$ (use this relation to eliminate $c_s$ from the result), and the wavelength of sound is much smaller than $s$ and $d$. The media to propagate sound moves together with M (i.e. at rest with respect to M).<br>Shortly before P and Q meet, the sound frequency that Q hears is $f_r$ . Calculate $f_r$.<br>在与第 D2 题相同设定下，观测者 P（从时间足够早开始）持续向所有方向发出声波，声源的振动频率为 $f$，声波的速度 $c_s$ 满足 $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$ （用此关系在结果中消去 $c_s$），声波的波长远小于 $s$ 和 $d$。声波的传播介质跟随质点 M 运动（即与 M 相对静止）。<br>在 P 和 Q 即将相遇前，Q 听到的声音频率是 $f_r$。求 $f_r$。 | **4 points**<br>**4 分** |

Solution:

This is a Doppler effect problem. However, since both the emitter and the receiver move with respect to the media with an angle, we need to derive the corresponding formula instead of using the 1-dimensional Doppler formula.

Let the sound wave emission time be $t_e$, reception time be $t_r$ (which equals to $t$ as given in the question). Let $L(t) = v(t_m - t)$ be the distance of P (or Q) from their meeting point, as a function of t. See figure below: (Look at the black lines now. The blue lines are for Problem D4.)



From the cosine theorem: $L^2(t_e) + L^2(t_r) - 2L(t_e)L(t_r)\cos\delta = c_s^2(t_r - t_e)^2$ -- (*). (1p)

We can do two things from equation (*).

(a) Find relation between $t_e$ and $t_r$. For this purpose, it is convenient to rewrite the LHS of (*) as $L^2(t_e) + L^2(t_r) - 2L(t_e)L(t_r)\cos\delta = \frac{c_s^2}{v^2}\left(L(t_e) - L(t_r)\right)^2$ . From $\frac{c_s^2 - v^2\cos\delta}{c_s^2 - v^2} = \frac{13}{12}$, we get

$L(t_e) = \frac{3}{2}L(t_r)$, i.e. $t_e = (3t_r - t_m)2 = (3t_r - \frac{[s+d\cot(\frac{\delta}{2})]}{v})/2$.

(1p, we will not use $t_e$ below. But the students may get $t_e$ which also deserves 1p.)

(b) To get the Doppler effect, we need to find relation between small variations $\delta t_e$ and $\delta t_r$. Varying equation (*), noting $\delta L(t) = -v\delta t$, we get

$$\frac{\delta t_e}{\delta t_r} = \frac{c_s^2+v^2(2-3\cos\delta)}{c_s^2-v^2(3-2\cos\delta)} = \frac{3}{2}\,.\ \text{(1p)}$$

(In the above, we have used $\frac{c_s^2-v^2\cos\delta}{c_s^2-v^2} = \frac{13}{12}$ and $L(t_e) = \frac{3}{2}L(t_r)$ in the last equal sign.)

Finally, the frequency at reception is

$$f_r = \frac{\delta t_e}{\delta t_r} f = \frac{3}{2} f\,.\ \text{(1p)}$$

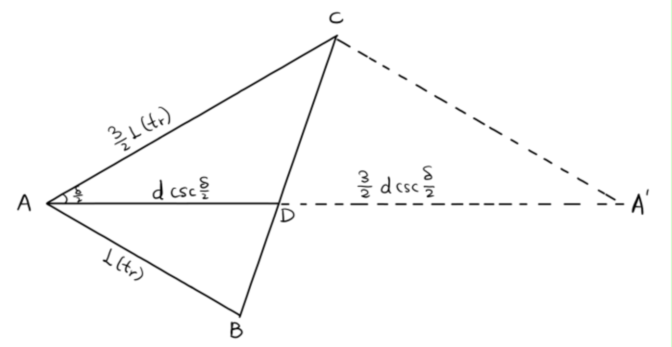
Alternatively, after getting $L(t_e) = \frac{3}{2}L(t_r)$, one may realize that the number of wave periods received is the same as the number of wave periods emitted, but in 2/3 time. Thus $f_r = \frac{3}{2}f$. This is also fully correct.

Note: if the student use 1D formula, i.e. by mistake considered the case where the media moves together with the middle of the PQ interval, then we give at most 2p out of 4p for the Doppler part: the relative velocity between the two points P and Q is $v_{\text{rel}} = 2v\sin\frac{\delta}{2}$ (1p), Corresponding frequency from Doppler effect, something like $f_r = \left(\frac{c_s+v_{\text{rel}}/2}{c_s-v_{\text{rel}}/2}\right)f$ , $f_r = \left(\frac{c_s+v_{\text{rel}}}{c_s}\right)f$ or $f_r = \left(\frac{c_s}{c_s-v_{\text{rel}}}\right)f$ (1p)

| | | |
|---|---|---|
| **D4** | Consider the same setup as Problem D3, starting from which time (i.e. find the corresponding $t$) on, Q starts to hear this frequency $f_r$?<br>在与 D3 题相同的设定下，从何时起（即计算其时间 $t$），Q 开始听到频率 $f_r$ ？ | **3 points**<br>**3 分** |

Solution:

For the frequency $f_r$ to appear, the sound has to be to the left of M (otherwise there is no deficit angle). This is illustrated as the blue lines in the above figure. Consider the marginal case that the sound crosses M. (1p)



The problem then is converted to the geometry problem above:

Given ∠BAD = ∠CAD = $\delta/2$, and |AB| = $L(t_r)$ = 3|AC|/2. Then what’s the relation between the lengths |AC| and |AD|? Draw line CA’ parallel to AB, and intersect the extension of AD at A’. Since ∠CA’D = $\frac{\delta}{2}$, |A’C| = |AC|. From the relation of similar triangles, |A’D| = 3|AD|/2. Thus, |AA’| = 5|AD|/2. Applying the cosine theorem:

$$\left(\frac{3}{2}L(t_r)\right)^2+\left(\frac{5}{2}d\csc\frac{\delta}{2}\right)^2-\left(\frac{3}{2}L(t_r)\right)\left(\frac{5}{2}d\csc\frac{\delta}{2}\right)\cos\frac{\delta}{2}=\left(\frac{3}{2}L(t_r)\right)^2 \quad \text{(1p)}$$

Solve this equation for $L(t_r)$, and apply the relation between $L(t_r)$ and $t_r$, we get

$L(t_r)=\frac{5d}{6\cos\frac{\delta}{2}\sin\frac{\delta}{2}}=\frac{5d}{3\sin\delta}$. And thus $t_r=\frac{s+d\cot\frac{\delta}{2}}{v}-\frac{5d}{3v\sin\delta}$.

Alternative approach:

$$\text{Area of ACB}=\frac{1}{2}\left(\frac{3}{2}L(t_r)\right)L(t_r)\sin\delta=\frac{3}{4}L(t_r)^2\sin\delta.$$

$$\text{Area of ACD}=\frac{1}{2}\left(\frac{3}{2}L(t_r)\right)\left(d\csc\frac{\delta}{2}\right)\sin\frac{\delta}{2}=\frac{3}{4}L(t_r)d.$$

$$\text{Area of ABD}=\frac{1}{2}L(t_r)\left(d\csc\frac{\delta}{2}\right)\sin\frac{\delta}{2}=\frac{1}{2}L(t_r)d.$$

Therefore,

$$\frac{3}{4}L(t_r)^2\sin\delta=\frac{3}{4}L(t_r)d+\frac{1}{2}L(t_r)d \quad\Rightarrow\quad L(t_r)=\frac{5d}{3\sin\delta}.$$

Using $L(t_r)=v(t_m-t)$, we obtain

$$t=\frac{s+d\cot\frac{\delta}{2}}{v}-\frac{5}{3\sin\delta}\frac{d}{v}.$$

Note: In reality, although we do not note a danger under dimensional attack to two dimensions, it is still meaningful to study the physics in two dimensions. For example, in our three-dimensional universe, there probably exist one-dimensional objects called “cosmic strings”. The cosmic strings in three dimensions are similar to point particles in two dimensions. Both of them bring a deficit angle to space. The three questions in Part D corresponds to the three important observable effects of cosmic strings. Searching for cosmic strings using these three observable effects is an active interdisciplinary research direction between high energy physics and astronomy.

注：现实中，尽管我们还没有发现被维数打击降为二维的风险，研究二维的物理仍然是有意义的。例如，我们的三维宇宙中，可能存在一种叫“宇宙弦”的线状一维物体。三维空间中的宇宙弦，和二维空间中的质点类似，都会给空间带来一个缺陷角。Part D 中的三个问题，对应的就是宇宙弦的三个重要观测效应。用这三个观测效应探测宇宙弦，是高能物理和宇宙学研究中的一个活跃的交叉领域。