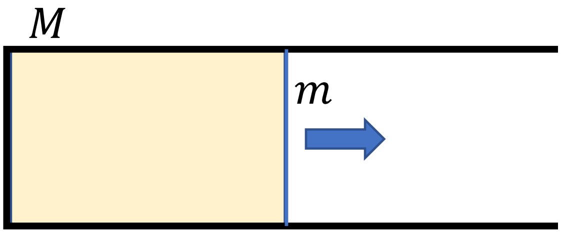
Pan Pearl River Delta Physics Olympiad 2020
2020 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 8 August, 2020)

Please fill in your final answers to all problems on the **answer sheet**. 请在**答题纸**上填上各题的最后答案。 At the end of the competition, please submit the **answer sheet only**. Question papers and working sheets will **not** be collected. 比赛结束时，请只交回**答题纸**，题目纸和草稿纸将**不会**收回。

1. [10 points]
A static insulating container of mass M and cylindrical shape is placed in vacuum. One of its ends is closed. Initially, an insulating piston of mass m and negligible width separates the volume of the container into two equal parts. The closed part contains n moles of monoatomic ideal gas with molar mass M_0 and temperature T. Assume that the container wall is smooth.
1. [10 分] 将质量为 M 的圆柱形绝缘容器静止置于真空中。其一端是封闭的。最初，质量 m 和宽度可忽略的绝缘活塞将容器分成两个相等的部分。封闭部分包含 n 摩尔的单原子理想气体，其温度为 T，摩尔质量为 M_0。假設容器為光滑容器。



(a) [2 points] Assume that the state of gas during its expansion can be approximated by thermal equilibrium condition, what is the temperature of the gas T_f when the piston left the container?
(a) [2 分] 假设气体在膨胀过程中的状态可以通过熱平衡条件来近似，求活塞离开容器时气体的温度 T_f 。

(b) [4 points] At the moment when the piston leaves the container, the gas and the container will move with speed v and the piston will move with speed u. Find v and u.
(b) [4 分] 在活塞离开容器的那一刻，气体和容器将以速率 v 移动，而活塞以速率 u 移动。求 v 和 u。
(c) [4 points] When all the gas has left the container, the final speed of the container further changes from v to $v + v'$. Estimate v' using the kinetic theory of gases. Assume that the final speed of the container is much less than the thermal speed of the molecules.
(c) [4 分] 当所有气体都离开容器后，容器的最终速度进一步从 v 变为 $v + v'$。使用气体动力学理论估算 v'。假设容器的最终速度远小于分子的热速度。

The gas constant is R. There is no heat exchange between the gas, container and the piston. The change of the temperature of the gas, when it leaves the container, can be neglected. The gravitation of the Earth can be neglected.
气体常数为 R。气体、容器和活塞之间没有热交换。气体离开容器后的温度变化可以忽略不计。可以忽略地球的引力。

Solution:
(a) When the gas expands, there is no heat exchange and the process is adiabatic. We have

$$TV^{\gamma-1} = T_f V_f^{\gamma-1}$$

where $\gamma = \frac{c_p}{c_v} = \frac{\frac{3}{2}+1}{\frac{3}{2}} = \frac{5}{3}$ and $V_f = 2V$. The final temperature is

$$T_f = 2^{-\frac{2}{3}}T$$

Reference: The validity range of the steady-state adiabatic law

Using the differential form of the first law of thermodynamics,

$$dU = \frac{3}{2}nRdT,$$

$$dQ = 0,$$

$$dW = pdV + Mvdv + \frac{1}{4}nM_0(v-u)(dv-du) + mudu.$$

This yields

$$\frac{3}{2}nRdT = -pdV - \left[\left(M + \frac{1}{4}nM_0\right)v - \frac{1}{4}nM_0u\right]dv - \left[\left(m + \frac{1}{4}nM_0\right)u - \frac{1}{4}nM_0v\right]du.$$

Let A be the cross-section area of the cylinder. Then in time dt,

$$dV = A(u+v)dt.$$

Here we have defined v in the forward direction, and u in the backward direction. Using conservation of linear momentum,

$$\left(M + \frac{1}{2}nM_0\right)v = \left(m + \frac{1}{2}nM_0\right)u.$$

Expressing u and v in terms of V,

$$u = \left(\frac{M + \frac{1}{2}nM_0}{M + m + nM_0}\right)\frac{1}{A}\frac{dV}{dt},$$

$$v = \left(\frac{m + \frac{1}{2}nM_0}{M + m + nM_0}\right)\frac{1}{A}\frac{dV}{dt}.$$

Eliminating u and v, and dividing by dt, we obtain

$$\frac{3}{2}nR\frac{dT}{dt} = -p\frac{dV}{dt}$$
$$-\left[\left(M+\frac{1}{4}nM_0\right)\left(m+\frac{1}{2}nM_0\right)^2 - \frac{1}{2}nM_0\left(M+\frac{1}{2}nM_0\right)\left(m+\frac{1}{2}nM_0\right) + \left(m+\frac{1}{4}nM_0\right)\left(M+\frac{1}{2}nM_0\right)^2\right]\frac{1}{(M+m+nM_0)^2}\frac{1}{A^2}\frac{dV}{dt}\frac{d^2V}{dt^2}.$$

Using ideal gas law,

$$\frac{3}{2}\frac{dT}{dt} + \frac{T}{V}\frac{dV}{dt} = -\frac{\mu}{nRA^2}\frac{dV}{dt}\frac{d^2V}{dt^2}.$$

Here we have introduced the reduced mass:

$$\mu = \left[\left(M+\frac{1}{4}nM_0\right)\left(m+\frac{1}{2}nM_0\right)^2 - \frac{1}{2}nM_0\left(M+\frac{1}{2}nM_0\right)\left(m+\frac{1}{2}nM_0\right) + \left(m+\frac{1}{4}nM_0\right)\left(M+\frac{1}{2}nM_0\right)^2\right]\frac{1}{(M+m+nM_0)^2}.$$

Let $x = TV^{\frac{2}{3}}$. Then we have

$$\frac{d(x^2)}{dt} = -\frac{4\mu}{3nRA^2}V^{\frac{2}{3}}\frac{dV}{dt}\frac{d^2V}{dt^2}.$$

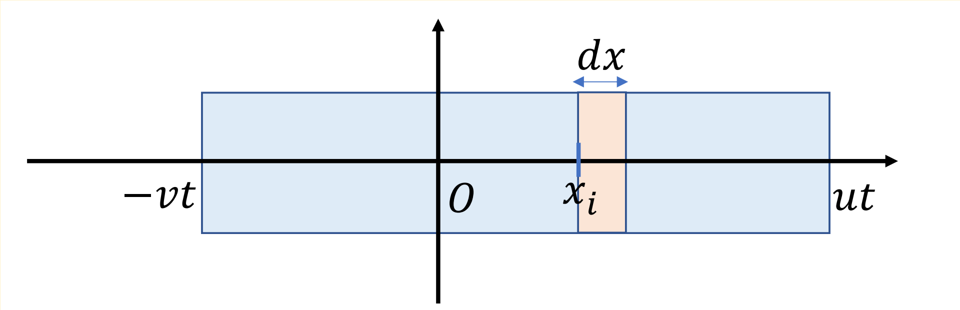
Although this equation is difficult to solve analytically, it shows that $TV^{\frac{2}{3}}$ is not a constant, that is, the steady-state adiabatic law is not applicable.

However, the steady-state adiabatic law is approximately correct if the motion during expansion is sufficiently low, since it is proportional to $\frac{1}{A}\frac{dV}{dt}$. In other words, the approximation is valid when the momentum of the system is negligible.

(b) By the conservation of momentum and energy,

$$(M+nM_0)v - mu = 0$$
$$\frac{1}{2}(M+nM_0)v^2 + \frac{1}{2}mu^2 = -\Delta U = \frac{3}{2}nR(T-T_f)$$
$$\Rightarrow v = \sqrt{3\left(1-2^{-\frac{2}{3}}\right)\frac{mnRT}{(nM_0+M)(m+nM_0+M)}}$$
$$\Rightarrow u = \sqrt{3\left(1-2^{-\frac{2}{3}}\right)\frac{(nM_0+M)nRT}{m(m+nM_0+M)}}$$

In fact, we can consider a more realistic situation. , If the thermal motion of the gas is much larger than v and u, we can assume that the density of gas ρ is uniform during the expansion.



Imagine that we have divided the gas into N slices, the mass of each slice is

$$\Delta m = \frac{nM_0}{N}$$

At time t, the location of the i-th slice is

$$x_i = -vt + \frac{(u+v)t}{N} i$$

for $i = 0,1,2,\dots,N-1$. Hence the velocity of the i-th slice is

$$\Rightarrow \dot{x}_i = -v + \frac{(u+v)}{N} i$$

The total momentum of the gas is

$$P_{gas} = \sum_i \Delta m \dot{x}_i = \frac{nM_0}{N} \sum_i \left(-v + \frac{u+v}{N} i\right) = nM_0 \left(-v + \frac{u+v}{N^2} \sum i\right)$$
$$= nM_0 \left(-v + \frac{u+v}{N^2} \frac{N(N+1)}{2}\right) \approx nM_0 \left(-v + \frac{u+v}{2}\right) = nM_0 \left(\frac{u-v}{2}\right)$$

The total kinetic energy of the gas is,

$$E_{gas} = \sum_i \frac{1}{2} \Delta m \dot{x}_i^2 = \frac{nM_0}{2N} \sum_i \left(v^2 - \frac{2v(u+v)}{N} i + \frac{(u+v)^2}{N^2} i^2\right)$$
$$= \frac{nM_0}{2} \left(v^2 - \frac{2v(u+v)}{N^2} \frac{N(N+1)}{2} + \frac{(u+v)^2}{N^3} \frac{N(N+1)(2N+1)}{6}\right)$$
$$\approx \frac{nM_0}{2} \left(v^2 - v(u+v) + \frac{(u+v)^2}{3}\right) = \frac{nM_0}{6} (u^2 - uv + v^2)$$

The conservation of energy and momentum give, (we define $\alpha = \frac{1}{2} nM_0$)

$$Mv + \alpha(v-u) - mu = 0$$
$$\frac{1}{2} Mv^2 + \frac{\alpha}{3}(u^2 - uv + v^2) + \frac{1}{2} mu^2 = -\Delta U = \frac{3}{2} nR(T - T_f)$$
$$\Rightarrow v^2 = \frac{3}{2} nRT \left(1 - 2^{-\frac{2}{3}}\right) \times \frac{(m+\alpha)^2}{(3M+\alpha)(m+\alpha)^2 + (3m+2\alpha)(M+\alpha)^2 - 2\alpha(M+\alpha)(m+\alpha)}$$
$$\Rightarrow u^2 = \frac{3}{2} nRT \left(1 - 2^{-\frac{2}{3}}\right) \times \frac{(M+\alpha)^2}{(3M+\alpha)(m+\alpha)^2 + (3m+2\alpha)(M+\alpha)^2 - 2\alpha(M+\alpha)(m+\alpha)}$$

(c) When the piston leaves the container, half of the gas particle will leave the container without any effect. And the other half will hit the bottom of the container before leaving the container. Each atom gives the container a momentum

$$\Delta p = 2m\overline{v_x}$$

where $m = M_0/N_A$ is the mass of each atom. By the kinetic theory,

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT_f$$

and

$$\overline{v^2} = 3\overline{v_x^2}$$

We get

$$\overline{v_x} \approx \sqrt{\frac{1}{3}\overline{v^2}} = \sqrt{\frac{kT_f}{m}}$$

$$\Rightarrow \Delta p = 2\sqrt{mkT_f}$$

The total momentum received by the container (due to the half of the gas) is

$$p = \left(\frac{nN_A}{2}\right)\Delta p = nN_A\sqrt{mkT_f} = nN_A\sqrt{\frac{M_0}{N_A}\frac{RT_f}{N_A}} = n\sqrt{M_0RT_f} = 2^{-1/3}n\sqrt{M_0RT}$$

Notice that $nRT = NkT \Rightarrow k = \frac{n}{N}R = R/N_A$. And the gain of the velocity is

$$v' = \frac{p}{M} = 2^{-1/3}\frac{n\sqrt{M_0RT}}{M}$$

Alternative solution:

We first assume p(t) is decaying exponentially with the relaxation time as the time scale. This assumption is valid if the gas is allowed to diffuse in space freely, but due to the particular geometry of the cylinder, the gas can only diffuse out the cylinder at the end. This significantly modifies the exponential decay of the pressure.

Without solving the hydrodynamic equations, we adopt the following simplified picture:

At $t = 0$, the gas near the end ($x = L$) starts to diffuse out, whereas gases inside the cylinder remains at the initial pressure.

At $0 < t < \frac{L}{v_{rms}}$, gases within a distance $L - v_{rms}t$ starts to diffuse out, whereas gases deeper inside the cylinder remains at the initial pressure. The pressure is approximately

$$p(x,t) = p_f \quad \text{for } x < L - v_{rms}t,$$

$$p(x,t) = p_f\, exp\left[-\frac{x - L + v_{rms}t}{l}\right] \text{ for } x > L - v_{rms}t,$$

where l is mean free path of the gas.

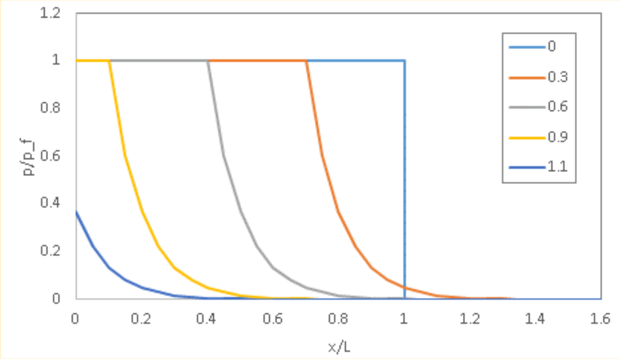
At $t > \frac{L}{v_{rms}}$, even gases near the end wall starts to diffuse out. The pressure is approximately

$$p(x,t) = p_f exp\left[-\frac{x - L + v_{rms}t}{l}\right].$$

Summarizing, we can write

$$p(x,t) = p_f \min\left\{1, exp\left[-\frac{x - L + v_{rms}t}{l}\right]\right\}.$$

The result is summarized in the following figure, where time is scaled in units of L/v_{rms}. For illustration, we have used $\frac{L}{l} = 10$, which is exaggerated.



Now we can focus on the pressure on the wall. The pressure is effectively unchanged before the time $\frac{L}{v_{rms}}$, and exponentially decay afterwards. At $x = 0$,

$$p(0,t) = p_f \quad \text{for } t < \frac{L}{v_{rms}},$$

$$p(0,t) = p_f exp\left[\frac{L - v_{rms}t}{l}\right] \quad \text{for } t > \frac{L}{v_{rms}}.$$

Hence the total momentum transfer to the wall is

$$\Delta p = A\,p_f\left(\frac{L}{v_{rms}}\right) + Ap_f\int_0^\infty exp\left(-\frac{t}{\tau}\right)dt,$$

where $\tau = \frac{l}{v_{rms}}$ is the relaxation time. The result is

$$\Delta p = Ap_f\frac{L}{v_{rms}} + Ap_f\tau. \qquad [1]$$

The first term is the result of kinetic theory, whereas the second term is due to the period of unchanged pressure. For $L >> l$, the first term is most important.

Interpretation of the second term

In eqtn [1], the physics of the first was originated from the kinetic theory of the molecules, which is valid at low density. The second term can be interpreted in the following way.

At higher density, a hydrodynamic model of the molecules is required. The mean free path l of the molecules is given by the condition

$$\rho\pi D^2 l = 1.$$

Here $\rho = nN_A/V_f$ is the number density of the molecules. N_A is the Avogadro's number. Hence

$$l = \frac{V_f}{\pi D^2 N_A n}.$$

The average speed of the molecules is given by

$$\sqrt{\langle v^2\rangle} = \sqrt{\frac{3k_BT_f}{m_0}} = \sqrt{\frac{3RT_f}{M_0}}.$$

Hence the relaxation time of gas density is given by

$$\tau = \frac{l}{\sqrt{\langle v^2 \rangle}} = \frac{V_f}{\pi D^2 N_A n}\sqrt{\frac{M_0}{3RT_f}}.$$

The momentum change of the container is given by the impulse, which is the integration of the force and the time. Hence the momentum change Δp is given by

$$\Delta p = Ap_f\tau = \frac{Ap_fV_f}{\pi D^2 N_A n}\sqrt{\frac{M_0}{3RT_f}}.$$

Using ideal gas law, $p_fV_f = nRT_f$,

$$\Delta p = Ap_f\tau = \frac{A}{\pi D^2 N_A}\frac{\sqrt{M_0 RT}}{3^{\frac{1}{2}}2^{\frac{1}{3}}}.$$

And the change of the velocity is

$$v' = \frac{A}{\pi D^2 N_A}\frac{\sqrt{M_0 RT}}{3^{1/2}2^{1/3}M}$$

2. **[10 points]** A spherical dust particle falls from rest through a water mist cloud of uniform density. The initial mass and radius of the spherical dust is M_0 and R_0 respectively. The rate of accretion onto the droplet is equal to the volume of the mist cloud swept out by the droplet per unit time. Let ρ be the density of water mist and g be the gravitational acceleration.
We assume the density of water mist ρ does not change after accretion and ignore air friction other than that from accretion.
2. [10 分] 球形尘埃粒子从静止的地方通过均匀密度的水雾云落下。球形尘埃的初始质量和半径分别为M_0 和R_0。液滴上的吸积率等于每单位时间被液滴扫出的雾状云的体积。设 ρ 为水雾的密度，g 为重力加速度。
我们假设水雾的密度 ρ 在吸积后没有变化，也忽略除吸积过程以外的空气摩擦。

(a) [2 points] Let $M(t)$ and $R(t)$ be the mass and radius of the droplet at time t respectively. Find a relationship between $\frac{dM}{dt}$ and $\frac{dR}{dt}$.
(a) [2 分] 设 $M(t)$ 和 $R(t)$ 分别为液滴在时间 t 的质量和半径。求 $\frac{dM}{dt}$ 和$\frac{dR}{dt}$ 之间的关系。

(b) [2 points] Find the speed of the droplet at time t. Express the answer in term of ρ, R and $\dot{R}$.
(b) [2 分] 求在时间 t 的液滴速度。用 ρ, R 和 $\dot{R}$ 表示答案。

(c) [3 points] After a long time, the radius of the droplet increases with time as $R(t) = bt^n$. Find b and n. Express the answers in terms of ρ and g.
(c) [3 分] 长时间后，液滴的半径随时间增加，满足关系式 $R(t) = bt^n$。求 b 和 n。用 ρ 和 g 表示答案。

(d) [3 points] Find the value of the acceleration of the droplet after a long time. Express the answer in terms of ρ and g.
(d) [3 分] 找出长时间后液滴的加速度值。用 ρ 和 g 表示答案。

Solution:
(a) Take the initial position of the dust particle as the origin and the x-axis along the downward vertical. Let $M(t)$ and $R(t)$ be the mass and radius of the droplet at time t respectively. Then

$$M(t) = M_0 + \frac{4}{3}\pi(R^3 - R_0^3)\rho$$

Where ρ is the density of the water mist.

$$\frac{dM}{dt} = 4\pi\rho R^2 \frac{dR}{dt}$$

(b) The droplet has a cross section πR^2 and sweeps out a cylinder of volume $\pi R^2 \dot{x}$ in unit time. As the rate of accretion is proportional to this volume, we have

$$\frac{dM}{dt} = \rho\pi R^2 \dot{x}$$

Hence, the speed of the droplet is,

$$\dot{x} = 4\dot{R}$$

(c) The momentum conservation gives

$$M(t+dt)\dot{x}(t+dt) - M(t)\dot{x}(t) = Mgdt$$

$$\Rightarrow \frac{d}{dt}(M\dot{x}) = Mg$$

$$\Rightarrow \dot{x}\frac{dM}{dt} + M\ddot{x} = Mg$$

For large t, $M(t) \approx \frac{4}{3}\pi R^3\rho,\ \dot{x} = 4\rho\dot{R}$ and $\frac{dM}{dt} \approx \frac{3M\dot{R}}{R}$, we have

$$\ddot{R} + \frac{3\dot{R}^2}{R} = \frac{g}{4}$$

A particular solution for large t has the form

$$R(t) = bt^n$$

Substituting into the DE,

$$bn(n-1)t^{n-2} + \frac{3b^2n^2t^{2(n-1)}}{bt^n} = (bn(n-1) + 3bn^2)t^{n-2} = \frac{g}{4}$$

$$\Rightarrow n = 2$$

and

$$\Rightarrow b = \frac{g}{56}$$

$$\Rightarrow R(t) = \frac{g}{56}t^2$$

(d)

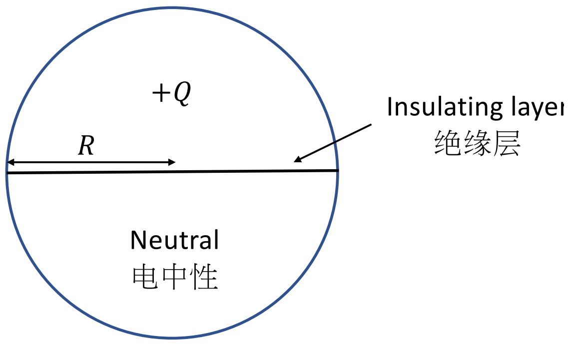
$$\dot{x} = 4\dot{R} = 4\left(\frac{g}{28}\right)t = \frac{g}{7}t$$

$$\Rightarrow \ddot{x} = \frac{g}{7}$$

The acceleration of the droplet is $g/7$ after a long time.

3. **[10 points]** There is a solid metallic sphere of radius R, which is cut into two identical hemispheres. The cut surface is coated with a thin insulating layer of thickness d, and the two parts are put together so that the original shape of the sphere is restored. Initially, the sphere is electrically neutral. Then one of the hemispheres is given a positive charge $+Q$ while the other one remains neutral. You can assume $d \ll R$ in this problem.

3. **[10 分]** 有一个半径为 R 的固体金属球，被切成两个相同的半球。切割表面涂上一层厚度为 d 的薄绝缘层，并将两部分放回一起，以恢复球形。最初，金属球是电中性的。然后，一个半球被赋予正电荷 $+Q$ ，而另一个则保持中性。在此问题中可假设 $d \ll R$。



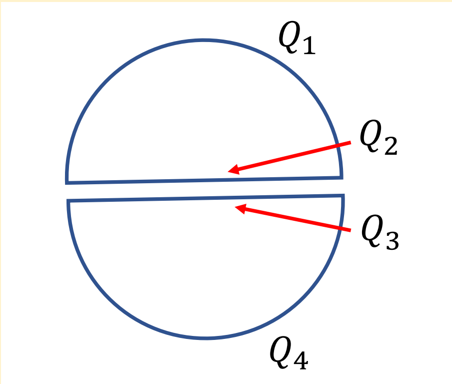
(a) [3 points] Find the charge on each surfaces of the sphere.
(a) [3 分] 找出球体每个表面上的电荷。

(b) [4 points] Find the electrostatic interaction force between two hemispheres.
(b) [4 分] 找出两个半球之间的静电相互作用力。

(c) [3 points] Find the electrostatic energy of the sphere.
(c) [3 分] 求球体的静电能。

Solution:

(a) Inside the metallic hemispheres, the electric field should be zero. Next, consider a Gauss surface surrounding the insulating layer. The electric flux is zero and hence the net charge inside the close surface is zero. This means the two sides of the insulating layer have the same amount but opposite charge. And the insulating layer is negligibly thin, that means the charge doesn't contribute any electric field except for the small space between the charge, just like two infinite-large parallel conductors.
The electric field is zero inside the conducting sphere. This can only be done if the charge uniformly distributes on the outer surface.



We have the conditions:

$$Q_1 = Q_4$$
$$Q_1 + Q_2 = Q$$
$$Q_2 = -Q_3$$
$$Q_3 + Q_4 = 0$$

We get

$$Q_1 = Q_2 = Q_4 = \frac{Q}{2}$$
$$Q_3 = -\frac{Q}{2}$$

And the charge distribution are

$$\sigma_1 = \sigma_4 = \frac{Q}{4\pi R^2}$$

$$\sigma_2 = \frac{Q}{2\pi R^2}$$
$$\sigma_3 = -\frac{Q}{2\pi R^2}$$

(b) The electric pressure (electric force per unit area) on the surface of a conductor is given by

$$P = \sigma E_{ext} = \frac{\sigma^2}{2\epsilon_0}$$

The net force acting on the upper hemisphere is

$$F_{upper} = (P_1 - P_2)\pi R^2 = \frac{\pi R^2}{2\epsilon_0}(\sigma_1^2 - \sigma_2^2) = \frac{\pi R^2}{2\epsilon_0}\left(\frac{Q^2}{16\pi^2 R^4} - \frac{Q^2}{4\pi^2 R^4}\right) = \frac{Q^2}{2\epsilon_0 \pi R^2}\left(\frac{1}{16} - \frac{1}{4}\right)$$
$$= -\frac{3Q^2}{32\epsilon_0 \pi R^2}$$

The force is acting downward.

For the lower hemisphere is, we have

$$F_{lower} = (P_3 - P_4)\pi R^2 = \frac{3Q^2}{32\epsilon_0 \pi R^2}$$

which is acting upward. Hence the electrostatic (attractive) force between two hemispheres is $F = \frac{3Q^2}{32\epsilon_0 \pi R^2}$.

(c) We can calculate the electrostatic energy of the sphere using the electric field energy density $u = \frac{1}{2}\epsilon_0 E^2$.

Inside the insulating layer, the energy is

$$U_1 = \frac{1}{2}\epsilon_0\left(\frac{\sigma_2}{\epsilon_0}\right)^2 \times \pi R^2 d = \frac{Q^2}{8\epsilon_0 \pi R}\left(\frac{d}{R}\right)$$

where $d \ll R$ is the thickness of the layer.
Inside the conductor, the electric field is zero and there is no energy associated with the field.

$$U_2 = 0$$

Outside the sphere, the electric field is

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$$

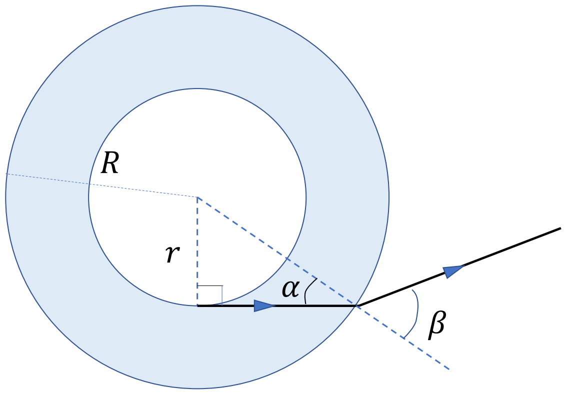
And the total energy is

$$U_3 = \frac{1}{2}\epsilon_0 \int \frac{Q^2}{16\pi^2\epsilon_0^2 r^4} 4\pi r^2 dr = \frac{Q^2}{8\pi\epsilon_0}\int_R^\infty \frac{1}{r^2} dr = \frac{Q^2}{8\pi\epsilon_0 R}$$

The total electrostatic energy of the sphere is

$$E = E_1 + E_2 + E_3 \approx \frac{Q^2}{8\pi\epsilon_0 R}$$

4. **[10 points]** Figure 4 shows a hollow glass tube with outer radius R and inner radius r respectively. The refractive index of the glass is n. From the outside air, the apparent inner radius of the tube is r'(i.e. the radius of the hollow portion observed from outside appears to be equal to r').
4. [10 分] 圖 4 表示一支中空玻璃管，其外半徑為 R，內半徑為 r，玻璃折射率為 n。由外面空氣中看來，該管之視內半徑為 r' (即空心部分的半徑從外面看起來等於 r')。



(a) [2 points] Find the ratio of the actual inner radius to the outer radius $\frac{r}{R}$. Express your answer in terms of α, β and n.
(a) [2 分] 求真內半徑與外半徑的比值 $\frac{r}{R}$，以α, β 和n来表示。

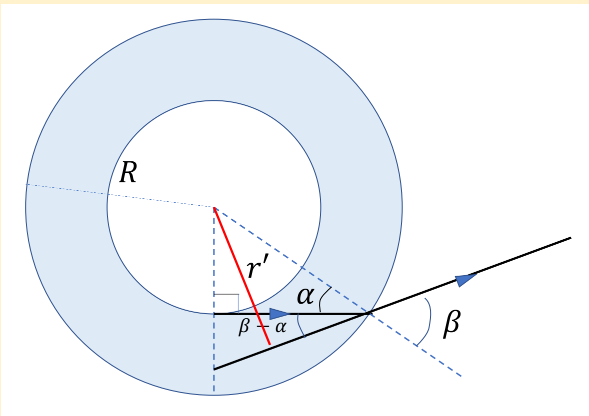
(b) [3 points] Find the ratio of the apparent inner radius to the outer radius $\frac{r'}{R}$. Express your answer in terms of α, β and n.
(b) [3 分] 求視內半徑與外半徑的比值 $\frac{r'}{R}$，以α, β 和n来表示。

(c) [5 points] If $R = 4.0$ mm, $r' = 0.50$ mm and $n = 1.6$, calculate the actual inner radius r of the glass tube up to 2 significant figures.
(c) [5 分] 若$R = 4.0$ mm，$r' = 0.50$ mm，$n = 1.6$，計算玻璃管的真內半徑r，答案準確至兩位有效數字。

Solution:
(a)

$$\frac{r}{R} = \sin\alpha$$

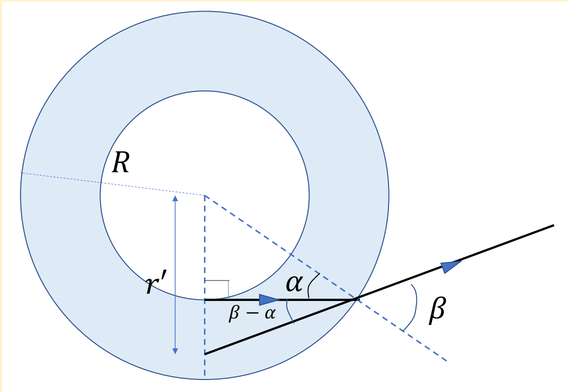
(b)



$$r' = R\sin\beta$$

$$\Rightarrow \frac{r'}{R} = \sin\beta = n\sin\alpha$$

Due to a different interpretation of the apparent radius, we also accept the following solution.



From the figure, we have

$$n \sin\alpha = \sin\beta$$

$$\frac{r'}{R} = \sin\alpha + \cos\alpha\tan(\beta-\alpha) = \frac{\sin\alpha\cos(\beta-\alpha)+\cos\alpha\sin(\beta-\alpha)}{\cos(\beta-\alpha)} = \frac{\sin\beta}{\cos(\beta-\alpha)} = \frac{n\sin\alpha}{\cos(\beta-\alpha)}$$

(c) It is given that

$$\frac{r'}{R} = 0.125 = 1.6\sin\alpha \Rightarrow \frac{r}{R} = \sin\alpha = \frac{0.125}{1.6} = 0.078125$$
$$\Rightarrow r = 0.31 \text{ mm}$$

According to the other answer in part (b), we have

$$\frac{r'}{R} = 0.125 = 1.6\frac{\sin\alpha}{\cos(\beta-\alpha)}$$
$$\Rightarrow \frac{\sin\alpha}{\cos(\beta-\alpha)} = \frac{0.125}{1.6} = 0.078125 = x \ll 1 \qquad [4.1]$$

Since $\sin\alpha = \frac{r}{R} < \frac{r'}{R} = 0.125$ and $\sin\beta = n\sin\alpha < 1.6\times 0.125 = 0.2$ are small, we can approximate

$$\frac{\sin\alpha}{\cos(\beta-\alpha)} \approx \frac{\alpha}{1-\frac{1}{2}(\beta-\alpha)^2} \approx \frac{\alpha}{1-\frac{1}{2}(n-1)\alpha^2} = x$$
$$\Rightarrow \frac{1}{2}(n-1)x\alpha^2 + \alpha - x = 0$$
$$\alpha = 0.07798$$
$$\Rightarrow \frac{r}{R} = 0.07798$$
$$r = 0.31 \text{ mm}$$

If you solve the equation [4.1] exactly, you will get $\alpha = 0.07812$ and $r = 0.312$ mm