Pan Pearl River Delta Physics Olympiad 2021
2021 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 15th May 2021)

Please fill in your final answers to all problems on the **answer sheet**.
请在**答题纸**上填上各题的最后答案。
At the end of the competition, please submit the **answer sheet only**. Question papers and working sheets will **not** be collected.
比赛结束时，请只交回**答题纸**，题目纸和草稿纸将不会收回。

1. [10 points]
A uniform thin rigid rod of mass m is supported by two rapidly rotating rollers, whose axes are separated by a fixed distance a. The rod is initially placed at rest symmetrically as shown in Fig.1a.
质量為 m 的均匀细刚性杆由两个快速旋转的滚筒支撑，两个滚筒的轴线距离為 a。杆最初如图 1a 所示对称放置。

(a) [5 points] Assume that the rollers rotate in opposite directions as shown in Fig.1a. The coefficient of kinetic friction between the rod and the rollers is μ. Solve for the displacement $x(t)$ of the center C of the rod from roller 1 assuming $x(0) = x_0$ and $\dot{x}(0) = 0$.
(a) [5 分] 假设滚筒以相反方向旋转，如图 1a 所示。杆和滚筒之间的动摩擦系数为 μ。假设$x(0) = x_0$ 和 $\dot{x}(0) = 0$，求杆中心 C 从滚筒 1 量度的位移 $x(t)$。

(b) [5 points] Now consider the case in which the direction of rotation of the rollers are reversed, as shown in Fig.1b. Find the displacement $x(t)$, again assuming $x(0) = x_0$ and $\dot{x}(0) = 0$.
(b) [5 分] 现在考虑使滚筒的旋转方向反向的情况，如图 1b 所示。假设$x(0) = x_0$ 和 $\dot{x}(0) = 0$，求杆中心 C 从滚筒 1 量度的位移 $x(t)$。

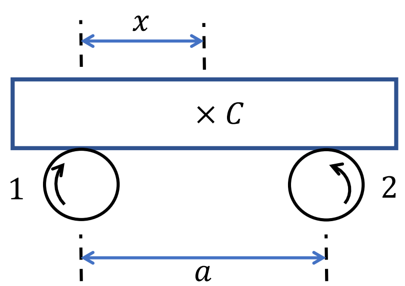


Fig. 1a

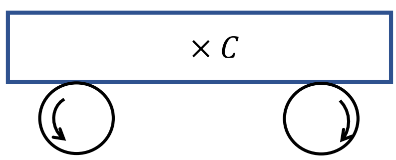
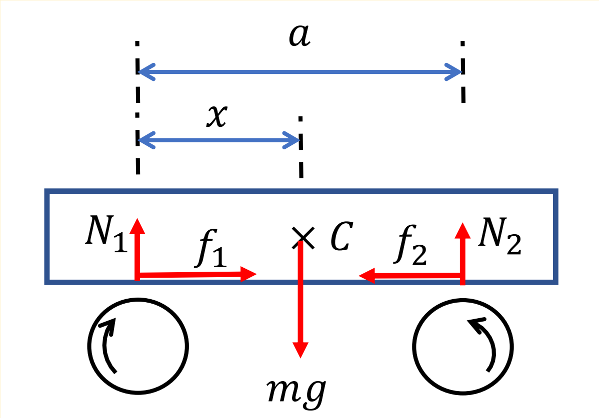


Fig. 1b

Solution:
(a) The free-body diagram of the rod is:



For equilibrium along the vertical direction, we have

$$N_1 + N_2 = mg$$
$$aN_2 = xmg$$
$$\Rightarrow N_1 = \left(1 - \frac{x}{a}\right) mg$$
$$\Rightarrow N_2 = \frac{x}{a} mg$$

The kinetic friction forces are

$$f_1 = \mu N_1 \quad and \quad f_2 = \mu N_2$$

With directions as shown in the figure.
Newton's 2^{nd} law gives

$$m\ddot{x} = f_1 - f_2 = \frac{\mu mg}{a}(a - 2x)$$

Define $\xi = 2x - a$, we have

$$\ddot{\xi} = -\frac{2\mu g}{a}\xi$$

is a SHM.

$$\Rightarrow \xi(t) = 2x(t) - a = 2A\cos(\omega t + \phi)$$

where $\omega = \sqrt{\frac{2\mu g}{a}}$.

$$\Rightarrow x(t) = A\cos(\omega t + \phi) + \frac{a}{2}$$

At $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = 0$, we have

$$x(t) = \left(x_0 - \frac{a}{2}\right)\cos\left(\sqrt{\frac{2\mu g}{a}}\,t\right) + \frac{a}{2}$$

(b) By reversing the direction of rotation of the rollers, we have

$$m\ddot{x} = f_2 - f_1 = \frac{\mu mg}{a}(2x - a)$$

Define $\xi = 2x - a$, we have

$$\ddot{\xi} = \frac{2\mu g}{a}\xi$$
$$\Rightarrow \xi(t) = 2Ae^{\omega t} + 2Be^{-\omega t}$$

where $\omega = \sqrt{\frac{2\mu g}{a}}$.

$$\Rightarrow x(t) = Ae^{\omega t} + Be^{-\omega t} + \frac{a}{2}$$

By the initial conditions, we have $A = B$ and

$$x(0) = x_0 = 2A + \frac{a}{2} \Rightarrow A = \frac{1}{2}\left(x_0 - \frac{a}{2}\right)$$

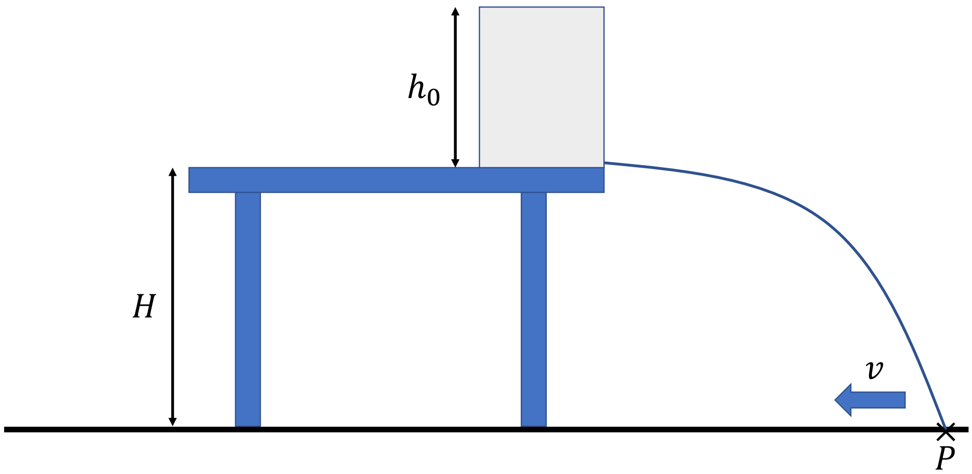
$$\Rightarrow x(t) = \left(x_0 - \frac{a}{2}\right)\frac{e^{\omega t} + e^{-\omega t}}{2} + \frac{a}{2} = \left(x_0 - \frac{a}{2}\right)\cosh \omega t + \frac{a}{2}$$

$$x(t) = \left(x_0 - \frac{a}{2}\right)\cosh\left(\sqrt{\frac{2\mu g}{a}}t\right) + \frac{a}{2}$$

2. [10 points]

A cylindrical tank filled with liquid is placed on the side of the table. A small hole is opened on the wall of the tank near the bottom. The liquid flows out horizontally through the small hole, and hits at point P on the floor. The height of the table is H, the area of the hole is $\frac{1}{k}$ (assume $k \gg 1$) of the area of the bottom of the tank, and the initial height of the liquid in the tank is h_0. Find the velocity v at which the drop point P moves along the floor and the time T required for all liquid to flow out of the tank.

在桌边放着装有液体的圆柱形容器，容器壁靠近底部开有小孔，液体经小孔水平流出，液柱射在地板上的P点。桌面高度为H，孔的面积是容器底部面积的$\frac{1}{k}$(假设 $k \gg 1$)，原来容器中液体高h_0。求落点P沿地板移动的速率 v 及所有液体从容器中流出所需的时间 T。



Solution:

(a) Assuming the when the height of the liquid inside the cylindrical tank is h, the drop of the water surface inside the tank has speed u and the liquid coming out from the hole has speed V. Since the volume of the liquid is unchanged, we have

$$V = ku$$

Bernoulli's equation gives

$$p_0 + \frac{1}{2}\rho V^2 = p_0 + \rho g h + \frac{1}{2}\rho u^2$$

$$\Rightarrow V = \frac{k\sqrt{2gh}}{\sqrt{k^2-1}}$$

The horizontal range of the liquid is

$$s = Vt = V \times \sqrt{\frac{2H}{g}} = \frac{k\sqrt{2gh}}{\sqrt{k^2-1}} \times \sqrt{\frac{2H}{g}} = \frac{2k\sqrt{hH}}{\sqrt{k^2-1}}$$

Therefore the position of P depends on the liquid height h. The speed of P is

$$v = \frac{ds}{dt} = \frac{2k\sqrt{H}}{\sqrt{k^2-1}}\frac{d}{dt}\sqrt{h} = \frac{k\sqrt{H}}{\sqrt{k^2-1}}\frac{1}{\sqrt{h}}\frac{dh}{dt} = \frac{k\sqrt{H}}{\sqrt{k^2-1}}\frac{u}{\sqrt{h}} = \frac{\sqrt{H}}{\sqrt{k^2-1}}\frac{V}{\sqrt{h}} = \frac{k}{k^2-1}\sqrt{2gH}$$

(b) There are 2 different ways to find T.

Method 1:

Initially, the position of P is

$$s_0 = \frac{2k\sqrt{h_0 H}}{\sqrt{k^2-1}}$$

Notice that v is independent of h, point P moves with the constant speed. When the tank is empty, P should be under the tank.

$$T = \frac{s_0}{v} = \frac{2k\sqrt{h_0 H}}{\sqrt{k^2-1}}\frac{k^2-1}{k}\frac{1}{\sqrt{2gH}} = \sqrt{\frac{2h_0(k^2-1)}{g}}$$

Method 2:

The speed of the water surface reads,

$$\frac{dh}{dt} = -u = -\frac{\sqrt{2gh}}{\sqrt{k^2-1}}$$

$$\Rightarrow \frac{dh}{\sqrt{h}} = -\sqrt{\frac{2g}{k^2-1}}dt$$

$$\Rightarrow 2\sqrt{h_0} = \sqrt{\frac{2g}{k^2-1}}T$$

$$\Rightarrow T = \sqrt{\frac{2h_0(k^2-1)}{g}}$$

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Remark:

The calculation above is based on the assumption that $k \gg 1$ and the flow is approximately steady (i.e. the velocity of the fluid inside the tank doesn't change in time). In fact, we can generalize the calculation by considering the non-steady flow of the fluid.

We first consider the potential energy of the fluid inside the tank when the height of the fluid is h,

$$U(h) = \rho g A h \times \frac{h}{2} \Rightarrow \frac{dU}{dt} = \rho g A h \frac{dh}{dt} = -\rho g h A u = -\rho g h \left(\frac{A}{k}\right) V$$

where u is the velocity of the fluid inside the tank and V is the velocity of the fluid coming out from the hole ($u = \frac{V}{k}$).

Next, the total kinetic energy of the fluid inside the tank is (N.B. because of the conservation of mass/continuity equation, the speed of the fluid inside the tank will be the same),

$$K_{tank} = \frac{\rho u^2}{2} A h = \frac{1}{2} \rho \frac{Ah}{k^2} V^2$$

$$\Rightarrow \frac{dK_{tank}}{dt} = \frac{1}{2} \frac{\rho A}{k^2} \frac{dh}{dt} V^2 + \rho \frac{Ah}{k^2} V \frac{dV}{dt} = -\frac{1}{2} \frac{\rho A}{k^3} V^3 + \frac{\rho A}{k^2} hV \frac{dV}{dt}$$

Finally, the rate at which kinetic energy exits the tank via the hole is

$$\frac{dK_{exit}}{dt} = \frac{dm_{exit}}{dt} \frac{V^2}{2} = \frac{\rho A}{k} \frac{V^3}{2}$$

Conservation of energy implies,

$$\frac{dU}{dt} + \frac{dK_{tank}}{dt} + \frac{dK_{exit}}{dt} = 0$$

$$\Rightarrow 2gh = \left(1 - \frac{1}{k^2}\right) V^2 + \frac{2}{k} h \frac{dV}{dt}$$

We can change the independent variable form t to h using the chain rule in calculus,

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = \frac{dV}{dh}(-u) = -\frac{V}{k} \frac{dV}{dh} = -\frac{1}{2k} \frac{dV^2}{dh}$$

We obtain the ODE,

$$2gh = \left(1 - \frac{1}{k^2}\right) V^2 - \frac{1}{k^2} h \frac{dV^2}{dh}$$

$$\Rightarrow \frac{dV^2}{dh} - (k^2 - 1) \frac{V^2}{h} = -2gk^2$$

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Introducing an integrating factor f such that

$$\frac{d}{dh}(fV^2) = f\frac{dV^2}{dh} + V^2\frac{df}{dh} = f\left(\frac{dV^2}{dh} - (k^2-1)\frac{V^2}{h}\right) = -2gk^2 f$$

$$\Rightarrow \frac{df}{dh} = -(k^2-1)\frac{f}{h}$$

$$\Rightarrow f = h^{1-k^2}$$

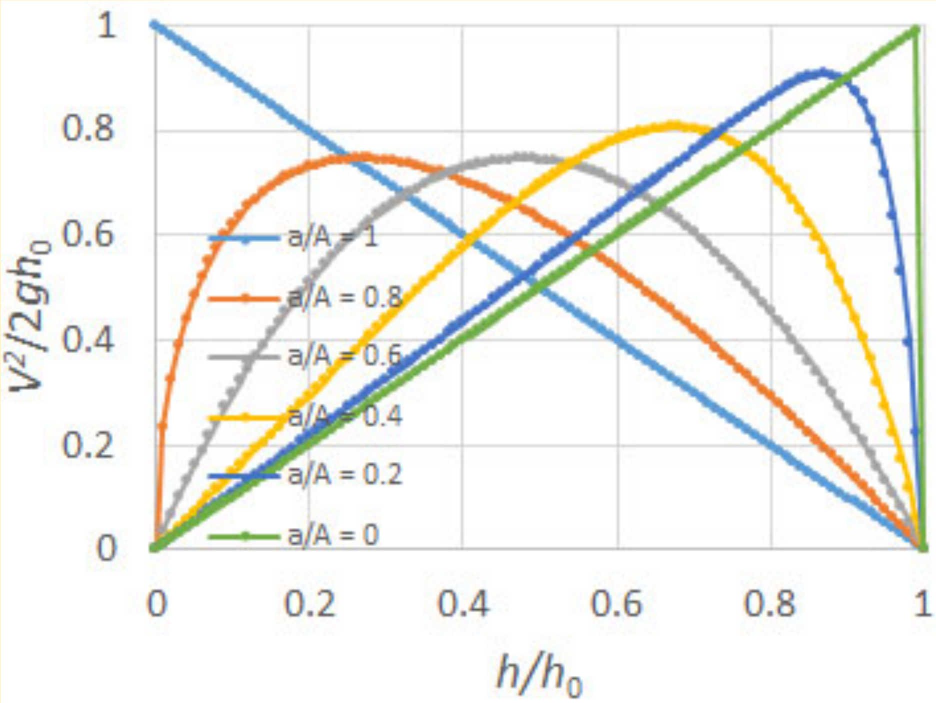
The ODE can be rewritten as

$$\frac{d}{dh}\left(h^{1-k^2}V^2\right) = -2gk^2h^{1-k^2}$$

Integrating the equation with the initial condition is that $V(h_0) = 0$, we have

$$V(h) = \sqrt{2gh}\sqrt{\frac{1-\left(\frac{h}{h_0}\right)^{k^2-2}}{1-\frac{2}{k^2}}}$$

The plot of the solution is shown below.



For the limiting case in which $k \to \infty$ (narrow opening) and $k \to 1$ (free fall of water), the solution reduces to

$$V(h, k=1) = \sqrt{2g(h_0 - h)}$$

$$V(h, k \to \infty) = \sqrt{2gh}$$

which is equal to our result obtained in part (a) in the limit of $k \to \infty$.

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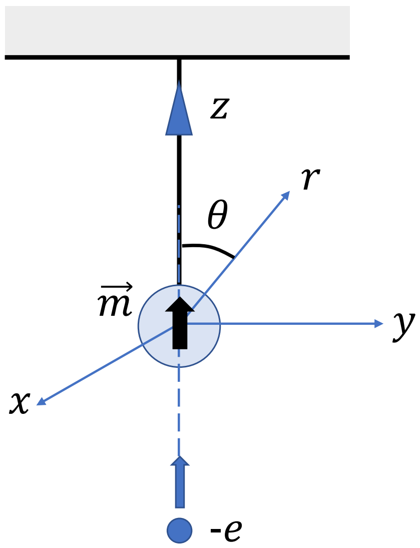
3. [10 points]
A uniformly magnetized iron sphere of magnetic moment $\vec{m} = m\hat{z}$ and radius R is suspended from the ceiling by an insulating thread. A total charge $Q > 0$ is uniformly distributed throughout the iron sphere. We use the Cartesian coordinate where the origin is located at the center of the sphere, xy-plane is the horizontal plane and z-axis is pointing upward.
磁矩为 $\vec{m} = m\hat{z}$ 且半径为 R 的均匀磁化铁球通过绝缘线悬挂在天花板上。总电荷$Q > 0$ 均匀分布在整个铁球中。我们使用笛卡尔坐标，其原点位于球体的中心，xy-平面是水平面，z轴指向上方。

The magnetic field at position r due to a magnetic dipole $\vec{m}$ at origin is given by
由位於原點的磁偶极子 $\vec{m}$ 所產生的磁场， 在位置 $\vec{r}$ 処为

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right)$$



(a) [3 points] The Poynting vector $\vec{S}(\vec{r})$ at point $\vec{r}$ is defined as
(a) [3 分] 坡印廷矢量$\vec{S}(\vec{r})$在点 $\vec{r}$ 上的定义为

$$\vec{S}(\vec{r}) = \frac{1}{\mu_0}\vec{E} \times \vec{B},$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields at point $\vec{r}$. Calculate the magnitude of the Poynting vector $\vec{S}(\vec{r})$ both inside and outside of the iron sphere.
其中 $\vec{E}$ 和 $\vec{B}$ 为点 $\vec{r}$ 的电场和磁场。计算铁球内部和外部的坡印廷矢量 $\vec{S}(\vec{r})$ 的大小。

(b) [2 points] In fact, the system has a non-zero angular momentum due to the electromagnetic field. The angular momentum L depends on the size of the sphere R, the charge Q and the magnetic dipole moment m on the sphere, and the physical constant μ_0 . We shall write $L = K\mu_0^\alpha R^\beta Q^\gamma m^\eta$ where K is a dimensionless constant. Find α, β, γ and η using dimensional analysis.
(b) [2 分] 实际上，由于电磁场的存在，系统的角动量非零。角动量 L 取决于铁球的半径 R，球上的电荷 Q 和磁偶极矩 m 以及物理常数 μ_0。我们将写成 $L = K\mu_0^\alpha R^\beta Q^\gamma m^\eta$ ，其中 K 是无量纲常数。使用量纲分析找 α，β，γ 和 η。

(c) [3 points] To calculate the total angular momentum $\vec{L}$ of the system, it is given than the angular momentum density [angular momentum of the EM field per unit volume] $\vec{l}(\vec{r})$ of the electromagnetic field at a point $\vec{r}$ is
(c) [3 分] 为了计算系统的总角动量 $\vec{L}$，已知在点 $\vec{r}$ 处的电磁场角动量密度 [电磁场在每单位体积的角动量] $\vec{l}(\vec{r})$ 是

$$\vec{l}(\vec{r}) = \frac{1}{c^2}\vec{r} \times \vec{S}$$

where c is the speed of light. Calculate the total angular momentum $\vec{L}$ of the system.
其中 c 是光速。计算系统的总角动量 $\vec{L}$。

(d) [2 points] Electrons are injected into the iron sphere along the z-axis. The total amount of the charge in the sphere will reduce and the sphere will rotate. Find the angular speed of the sphere after the injection of N electrons. Assume that the moment of inertia of the iron sphere is I and each electron has charge $-e$.
(d) [2 分] 电子沿 z 轴注入铁球。球体中的总电荷量减少，并且球体将旋转。找出 N粒电子注入后球的角速度。假设铁球的惯性矩为 I，并且每个电子的电荷为 $-e$。

Solution:

[Solution 1] Since iron is a conductor, the charges will redistribute on surface of the sphere.
(a) The electric field inside the sphere is zero and the electric field outside is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^3}\vec{r}$$

Hence the Poynting vector inside the sphere is $\vec{S} = 0$. Outside the sphere,

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

where the magnetic field of a magnetic dipole is,

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right)$$

$$\Rightarrow \vec{S} = \frac{Q}{4\pi\epsilon_0 r^3}\frac{1}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}\times\vec{r}}{r^2} - \vec{r}\times\vec{m}\right)$$

$$\Rightarrow \vec{S} = \frac{Qm}{16\pi^2\epsilon_0 r^5}(\hat{z}\times\hat{r})$$

$$\Rightarrow S = |\vec{S}| = \frac{Qm\sin\theta}{16\pi^2\epsilon_0 r^5}$$

(b) It is given that

$$L = K\mu_0^\alpha R^\beta Q^\gamma m^\eta$$

$$[L] = \frac{ML^2}{T}, [\mu_0] = \frac{ML}{Q^2}, [R] = L, [Q] = Q, [m] = \frac{QL^2}{T}$$

By dimensional analysis, we have

$$[Q]: -2\alpha + \gamma + \eta = 0$$

$$[M]: \alpha = 1$$

$$[L]: \alpha + \beta + 2\eta = 2$$

$$[T]: -\eta = -1$$

With 5 equations, we can get

$$\alpha = 1, \qquad \beta = -1, \qquad \gamma = 1, \qquad \eta = 1$$

And

$$L = K\frac{\mu_0 Qm}{R}$$

(c) The angular momentum density of the EM field is

$$\vec{l} = \vec{r} \times \frac{\vec{S}}{c^2} = \frac{\mu_0 Qm}{16\pi^2 r^4}\hat{r} \times (\hat{z} \times \hat{r}) = -\frac{\mu_0 Qm\sin\theta}{16\pi^2 r^4}\hat{e}_\theta$$

Because of symmetry, the total angular momentum has only the z-component, with magnitude

$$L_z = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \int_R^\infty r^2 dr \left(\frac{\mu_0 Qm\sin\theta}{16\pi^2 r^4}\right)\sin\theta = \frac{\mu_0 Qm}{16\pi^2}\int_0^{2\pi} d\phi \int_0^\pi \sin^3\theta\, d\theta \int_R^\infty \frac{1}{r^2}dr$$

$$\int_0^\pi \sin^3\theta\, d\theta = -\int_0^\pi (1-\cos^2\theta)d\cos\theta = -\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^\pi = -\left(-2 + \frac{2}{3}\right) = \frac{4}{3}$$

$$\Rightarrow \vec{L} = \frac{\mu_0 Qm}{6\pi R}\hat{z}$$

(c) As the electrons are being injected on the sphere, the charge Q decreases, causing the electromagnetic angular momentum decreases. By the conservation of angular momentum, we have

$$\frac{\mu_0 Qm}{6\pi R} = I\omega + \frac{\mu_0(Q - Ne)m}{6\pi R}$$

$$\Rightarrow \omega = \frac{\mu_0 Nem}{6\pi RI}$$

The angular speed of the sphere is $\omega = \frac{\mu_0 Nem}{6\pi RI}$

[Solution 2] In an alternative interpretation of the question, the charges are assumed to be distributed uniformly inside the entire sphere.

(a) The electric field and magnetic field inside the sphere are

$$\vec{E} = \frac{Q\vec{r}}{4\pi\epsilon_0 R^3}$$

$$\vec{B} = \frac{2}{3}\mu_0\vec{M} = \frac{2}{3}\mu_0\frac{\vec{m}}{\frac{4}{3}\pi R^3} = \frac{\mu_0 m}{2\pi R^3}\hat{z}$$

(The derivation of the $\vec{B}$ field inside the magnetized sphere is in the end).

The Poynting vector inside the sphere becomes,

$$\vec{S}_{in} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{Qmr}{8\pi^2\epsilon_0 R^6}(\hat{r}\times\hat{z})$$

(c) The total angular momentum consists of 2 parts,

$$\vec{L} = \vec{L}_{out} + \vec{L}_{in},$$

where the angular momentum outside the sphere is

$$\vec{L}_{out} = \frac{\mu_0 Qm}{6\pi R}\hat{z}$$

Inside the sphere, we have

$$\vec{l} = \vec{r}\times\frac{\vec{S}}{c^2} = \frac{\mu_0 Qmr^2}{8\pi^2 R^6}\hat{r}\times(\hat{r}\times\hat{z}) = \frac{\mu_0 Qmr^2\sin\theta}{8\pi^2 R^6}\hat{e}_\theta$$

$$\vec{L}_{in} = \int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\int_R^{\infty}r^2dr\left(-\frac{\mu_0 Qmr^2\sin\theta}{8\pi^2 R^6}\right)\sin\theta = -\frac{\mu_0 Qm}{15\pi R}\hat{z}$$

$$\Rightarrow \vec{L}_{tot} = \frac{\mu_0 Qm}{6\pi R}\hat{z} - \frac{\mu_0 Qm}{15\pi R}\hat{z} = \frac{\mu_0 Qm}{10\pi R}\hat{z}$$

(d) As the electrons are being injected on the sphere, the charge Q decreases, causing the electromagnetic angular momentum decreases. By the conservation of angular momentum, we have

$$\omega = \frac{\mu_0 Nem}{10\pi RI}$$

Appendix: Derivation of $\vec{B}$ field inside a uniformly magnetized sphere.

Let's consider a static magnetized sphere without electric current. Ampere's law gives

$$\vec{\nabla} \times \vec{H} = 0,$$

where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ is the auxiliary field. Mathematically, we know that a curl-free field can be rewritten as a gradient of a scaler field ϕ_m,

$$\vec{H} = -\vec{\nabla}\phi_m,$$

Where ϕ_m is called the magnetic scaler potential.

$$\vec{\nabla} \cdot \vec{B} = \mu_0 \vec{\nabla} \cdot \left(\vec{H} + \vec{M}\right) = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{H} = -\nabla^2 \phi_m = -\vec{\nabla} \cdot \vec{M} = \rho_m = 0. \text{ (inside the sphere)} \quad [1]$$

where $\rho_m = -\vec{\nabla} \cdot \vec{M}$ is the magnetic charge inside the sphere. However, there is a magnetic surface charge on the surface of the sphere, reads,

$$\sigma_m = \hat{r} \cdot \vec{M} = M \cos\theta.$$

Here, r and θ are sphereical coordinates. One of the boundary conditions at the surface is that the tangential component of $\vec{H}$ must be continuous,

$$\Rightarrow \phi_m(r = R^+, \theta) = \phi_m(r = R^-, \theta) \quad [2]$$

And the Gauss' law of magnetic charge gives,

$$\left.\frac{\partial \phi_m}{\partial r}\right|_{r=R^+} - \left.\frac{\partial \phi_m}{\partial r}\right|_{r=R^-} = -\sigma_m = -M\cos\theta \quad [3]$$

In other words, the magnetic charge on the surface of the sphere gives rise to a discontinuity in the radial gradient of the magnetic scaler potential at $r = R$.

The Laplace equation $\nabla^2 \phi_m = 0$ can be written in spherical coordinates,

$$\nabla^2 \phi_m = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\phi_m\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi_m}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\phi_m = 0$$

Since the boundary conditions are independent of φ, we expect $\phi_m = \phi_m(r, \theta)$ is independent of φ.

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\phi_m\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi_m}{\partial \theta}\right) = 0$$

Next, we apply the separation of variables $\phi_m(r, \theta) = A(r)B(\theta)$, we have

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$$\frac{d}{dr}\left(r^2\frac{d}{dr}A(r)\right) = \lambda A(r) \;\; and \;\; \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dB(\theta)}{d\theta}\right) = -\lambda B(\theta)$$

$$\Rightarrow r^2A''(r) + 2rA'(r) - \lambda A(r) = 0$$

Substitute $A(r) = r^n$,

The general solution of the Laplace's equation inside and outside the sphere can be written as

$$\phi_m^-(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad for\; r \le R$$

and

$$\phi_m^+(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad for\; r \ge R$$

where $\{A_l, B_l\}$ are constants can be determined by the boundary conditions and $P_l(\cos\theta)$ is the Legendre polynomials.

By the boundary conditions [2] and [3], we have

$$B_l = A_l R^{2l+1},$$

$$-\frac{2B_1}{R^3} - A_1 = 0,$$

$$-\frac{(l+1)B_l}{R^{l+2}} - lA_lR^{l-1} = 0. \text{ if } l \neq 1$$

Solving the algebraic equations, we have

$$A_l = B_l = 0. \quad \text{for } l \neq 1$$

and

$$\Rightarrow B_1 = \frac{MR^3}{3} \;\; and\; A_1 = \frac{M}{3}$$

The scaler potentials are:

$$\phi_m^-(r,\theta) = \frac{Mr}{3}\cos\theta$$

$$\phi_m^+(r,\theta) = \frac{MR^3}{3r^2}\cos\theta$$

In the vacuum region outside the sphere, the magnetic field is

$$\vec{B} = \mu_0\vec{H} = -\mu_0\vec{\nabla}\phi_m^+ = \frac{\mu_0}{4\pi}\left(-\frac{\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right)$$

where $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$. Inside the magnetized sphere, $\vec{H} = -\vec{\nabla}\phi_m$ and $\vec{B} = \mu_0(\vec{H} + \vec{M})$

$$\Rightarrow \vec{H} = -\frac{\vec{M}}{3}$$

And

$$\vec{B} = \frac{2}{3}\mu_0\vec{M}$$

4. **Air Convection in Atmosphere 大气内之空气对流**

(a) [1 point] Consider a horizontal slab of air whose thickness (height) is dz. If this slab is at rest, the pressure holding it up from below must balance both the pressure from above and the weight of the slab. Use this fact to find an expression for $\frac{dP}{dz}$, the variation of pressure with altitude, in terms of the density ρ of air.

(a) [1 分] 考虑一薄块的空气，其厚度(高度)是dz。当这薄块处于静止状态时，从下方施加于薄块的压强必须平衡于从上面施加于薄块的压强和薄块的自身重量。由此,找出压强随高度变化的表达式 $\frac{dP}{dz}$，答案以空气密度 ρ 来表示。

(b) [2 points] Assume that the air is an ideal gas with molar mass M and the temperature T of the atmosphere is independent of height. Then the atmospheric pressure at height z is given by $P(z) = P(0)e^{-\lambda z}$. Find λ.

(b) [2 分] 假设空氣是摩尔质量 M 的理想氣體，而且大气的温度 T 随高度无关。因此，在高度为 z 的大气压强可以由 $P(z) = P(0)e^{-\lambda z}$ 表示。 求λ。

In practice, the atmospheric temperature depends on height. If the temperature gradient $\left|\frac{dT}{dz}\right|$ exceeds a certain critical value, convection will occur: warm, low-density air will rise, while cool, high-density air sinks. The decrease of pressure with altitude causes a rising air mass to expand adiabatically and thus to cool. The condition for convection to occur is that the rising air mass must remain warmer than the surrounding air despite this adiabatic cooling.

在实际情况下，大气的温度会随高度变化。当温度梯度 $\left|\frac{dT}{dz}\right|$ 超越一个临界值时，对流就会产生:低密度的热空气上升，高密度的冷空气则下降。随高度上升而下降的气压，使上升的空气团发生绝热膨胀，从而冷却。对流发生的条件是:上升中的气团纵然发生绝热冷却，仍须较周围的空气温暖。

(c) [2 points] Assume that the molar heat capacity of air at constant volume is $c_V = \frac{5}{2}R$. We can show that when air expands adiabatically, the temperature and pressure are related by the condition

$$\frac{dT}{dP} = a\frac{T}{P}$$

Find the constant a.

(c) [2 分] 假设空气的定容摩尔热容量为$c_V = \frac{5}{2}R$。由此可证明，当空气绝热膨胀时，温度和压強满足以下条件

$$\frac{dT}{dP} = a\frac{T}{P}$$

求常数a。

(d) [3 points] Assume that $\frac{dT}{dz}$ is just at the critical value for convection to begin, so that the temperature drop due to adiabatic expansion of the convecting air mass is the same as the temperature gradient of the surrounding air. Find a formula for $\frac{dT}{dz}$ in this case.

(d) [3 分] 假设 $\frac{dT}{dz}$ 正处于对流开始发生的临界值，以致对流空气由绝热膨胀引起的温度下降，等于周边空气的温度梯度。在此情况下,求 $\frac{dT}{dz}$ 的公式。

(e) [2 points] Calculate numerically the critical temperature gradient in part (d). Express your answer in K/km.

Data: The molar mass of the air is $M = 0.029\ kg, g = 9.8\ \mathrm{m/s^2}, R = 8.31$ J/mol/K, $T = 300$K.

(e) [2 分] 计算(d)部的临界温度梯度之数值。答案以 K/km 为单位。数值: $M = 0.029$ kg, $g = 9.8\ \mathrm{m/s^2}, R = 8.31$ J/mol/K, $T = 300$K。

Solution:

(a) Consider a horizontal slab of air whose thickness (height) is dz and the cross-sectional area is A. In equilibrium, we have

$$P(z)A = \rho gAdz + P(z + dz)A \Rightarrow \frac{dP}{dz} = -\rho g$$

(b) Using the ideal gas law, $PV = nRT$,

$$P = \left(\frac{m}{V}\right)\frac{RT}{M} \Rightarrow \rho = \frac{MP}{RT}$$

Where M is the molar mass, m is the total mass and n is the number of mole of the gas.

$$\Rightarrow \frac{dP}{dz} = -\frac{Mg}{RT}P$$

$$\Rightarrow P(z) = P(0)e^{-\lambda z}$$

1

Where $\lambda = \frac{Mg}{RT}$.

(c) Using the 1st law,

$$dU = dQ - dW$$

For 1 mole of gas during the adiabatic expansion, $dU = c_V dT = \frac{5}{2} RdT$, $dQ = 0$ and $dW = PdV$

$$\Rightarrow \frac{5}{2} RdT = -PdV$$

Ideal gas law $PV = RT$ implies

$$PdV + VdP = RdT$$

$$\Rightarrow -\frac{5}{2} RdT + VdP = RdT$$

$$\Rightarrow \frac{7}{2} RdT = VdP$$

$$\Rightarrow \frac{dT}{dP} = \frac{2V}{7R} = \frac{2}{7}\frac{T}{P}$$

Therefore, $a = \frac{2}{7}$.

(d) Finally,

$$\frac{dT}{dz} = \left(\frac{dT}{dP}\right)\left(\frac{dP}{dz}\right) = \frac{2}{7}\frac{T}{P}\left(-\frac{Mg}{RT}P\right) = -\frac{2Mg}{7R}$$

(e) Critical temperature gradient is

$$\frac{dT}{dz} = -\frac{2Mg}{7R} = -9.8 \text{ K/km}$$