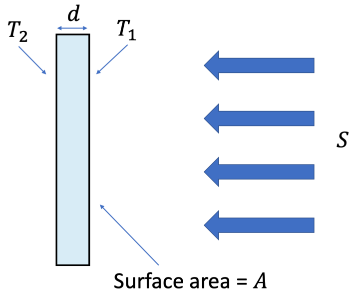
**Pan Pearl River Delta Physics Olympiad 2022**
**2022 年泛珠三角及中华名校物理奥林匹克邀请赛**
**Sponsored by Institute for Advanced Study, HKUST**
**香港科技大学高等研究院赞助**
**Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）**
**(9:30 am – 12:00 pm, 29th Jan 2023)**

Please fill in your final answers to all problems on the answer sheet.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. [10 points] Consider a satellite that has a shape of a plate of surface area $A$ and thickness $d \ll \sqrt{A}$. The satellite can convert solar energy into electrical energy and charge the onboard batteries making use of the temperature difference. The solar energy flux density [solar power per unit area] at the position of the satellite is $S$ and the satellite is facing towards Sun. Assuming that the emissivity of both sides of the satellite is $\epsilon$ and the temperatures of the satellite on two sides are $T_1$ and $T_2$ respectively and $\sigma$ is the Stefan-Boltzmann constant.
1. [10 分] 考虑一个人造卫星，其形状为表面积为 $A$ 且厚度为$d \ll \sqrt{A}$ 的平板。卫星可以利用温差将太阳能转化为电能，为星载电池充电。人造卫星所在位置的太阳能通量密度[单位面积的太阳能功率]为$S$，卫星正对着太阳。假设整颗卫星的发射率（又称辐射率）为$\epsilon$，卫星两侧的温度分别为 $T_1$ 和 $T_2$ ，$\sigma$ 是 Stefan-Boltzmann 常数。




(a) [2] What is the net heat flux [energy per second] **absorbed** by the bright side (the side facing towards Sun) of the satellite?
(a) [2] 卫星的亮面（面向太阳的一面）吸收的净热通量 [每秒能量] 是多少？
(b) [1] What is the net heat flux [energy per second] **released** from the dark side of the satellite?
(b) [1] 从卫星暗面**释放**的净热通量 [每秒能量] 是多少？
(c) [1] What is value of the emissivity $\epsilon$ to get the theoretically maximal charging power $P_{max}$?
(c) [1] 可得到理论上最大充电功率 $P_{max}$ 的发射率 $\epsilon$ 的值是多少？
(d) [3] Find a condition for the temperature $T_1$ in order to get the theoretically maximal charging power $P_{max}$ provided by the satellite. Express the condition in term of the dimensionless variable $x = \frac{\sigma T_1^4}{S}$. **You don't need to solve the equation in this part.**
(d) [3] 求温度$T_1$的条件，使得卫星能達到理论上最大充电功率 $P_{max}$。 用无量纲变量 $x = \frac{\sigma T_1^4}{S}$ 表示你的答案。 你不需要解这部分的方程。
(e) [3] What is the theoretically maximal charging power $P_{\max}$ provided by the satellite? Calculate the numerical value of $\frac{P_{\max}}{AS}$. Your answer should be correct to at least 2 significant figures.
(e) [3] 卫星可提供的理论上最大充电功率$P_{max}$是多少？ 计算 $\frac{P_{\max}}{AS}$ 的数值。 你的答案至少应正确到 2 位有效数字。

Solution:
(a)
$$\dot{Q}_1 = \epsilon A(S - \sigma T_1^4)$$

(b)

$$\dot{Q}_2 = \epsilon\sigma A T_2^4$$

(c) The power is given by

$$P = \dot{Q}_1 - \dot{Q}_2 = \epsilon A(S - \sigma T_1^4 - \sigma T_2^4) \qquad (1)$$

For maximal power, the satellite should be a Carnot engine,

$$\frac{\dot{Q}_1}{T_1} = \frac{\dot{Q}_2}{T_2}$$

$$\Rightarrow \frac{\epsilon A(S - \sigma T_1^4)}{T_1} = \frac{\epsilon\sigma A T_2^4}{T_2}$$

$$\Rightarrow T_2^3 = \frac{S}{\sigma T_1} - T_1^3$$

Sub. Into eqtn (1),

$$P = \epsilon A(S - \sigma T_1^4 - \sigma T_2^4) = \epsilon A\left(S - \sigma T_1^4 - \sigma\left(\frac{S}{\sigma T_1} - T_1^3\right)^{4/3}\right)$$

Obviously, we can get maximal power if $\epsilon = 1$.

(d) Define $x = \frac{\sigma T_1^4}{S}$, we have

$$\frac{P_{max}}{AS} = 1 - x - \left(\frac{1-x}{x^{\frac{1}{4}}}\right)^{\frac{4}{3}} = 1 - x - \frac{(1-x)^{4/3}}{x^{1/3}}$$

We can get max. power $P$ if $x$ satisfies the following equation:

$$\frac{d}{dx}\left(\frac{P_{max}}{AS}\right) = -1 - \frac{4}{3}\frac{(1-x)^{\frac{1}{3}}}{x^{1/3}} + \frac{(1-x)^{\frac{4}{3}}}{3x^{4/3}} = 0$$

(e) We can solve the equation numerically and we get:

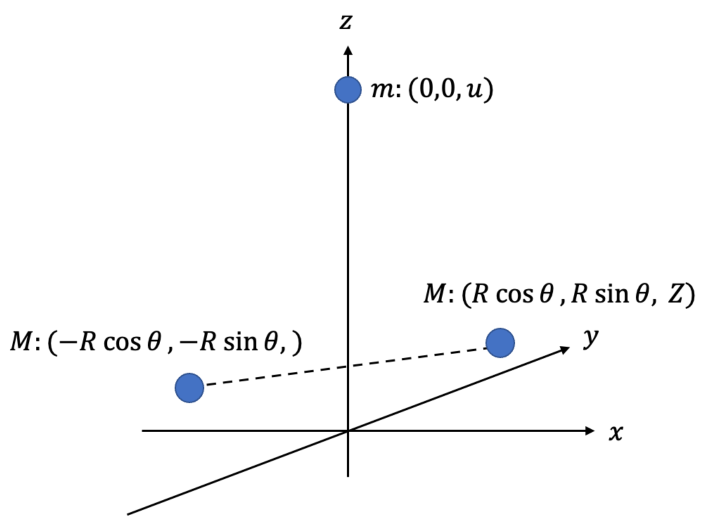
$$x \approx 0.751$$

The corresponding value of power is

$$\frac{P_{max}}{AS} = 0.077$$

2. [10 points] Binary-sun solar system: Consider a binary pair of identical suns of mass $M$ orbiting in the $x-y$ plane in an orbit centred at the origin. The gravitational constant is $G$. Now add a planet of mass $m$ with an initial condition on the $z$ axis above the center of mass of the two suns and with a velocity along the $z$ direction. By the symmetry of the system, the small planet will remain on the $z$ axis, suns will have equal $z$ coordinates and the center of mass of two suns will also remain on the $z$ axis.

2. [10 分]双星太阳系：考虑两个质量均为 $M$ 的相同太阳在 $x-y$ 平面中以原点为中心的轨道运行。万有引力常数为 $G$。现在添加一个质量 $m$ 的行星，其初始条件位于两个太阳质心上方的 z 轴上，速度在 z 方向上。通过系统的对称性，小行星将保持在 z 轴上，两个太阳的 z 坐标相同，其质心也维持在 z 轴上。




We use the Cartesian coordinates to describe the dynamics of the system: the coordinate of the planet $(0,0,u)$, and the coordinates of two suns are $(\pm R\cos\theta, \pm R\sin\theta, Z)$.

我们用笛卡尔坐标来描述系统的动力学：行星坐标为$(0,0,u)$，两个太阳的坐标为$(\pm R\cos\theta, \pm R\sin\theta, Z)$。

(a) [0.5] What is the total kinetic energy, $T$, of the system? Express the answer in terms of $R, \theta, Z, u$ and their time derivative.

(a) [0.5] 系统的总动能 $T$ 是多少？用 $R, \theta, Z, u$ 及其时间导数表示答案。

(b) [0.5] What is the total potential energy, $V$, of the system? Express the answer in terms of $R, \theta, Z, u$ and their time derivative.

(b) [0.5] 系统的总势能 $V$ 是多少？用 $R, \theta, Z, u$ 及其时间导数表示答案。

(c) [0.5] What is the total linear momentum, $P$, of the system? Express the answer in terms of $R, \theta, Z, u$ and their time derivative.

(c) [0.5] 系统的总线性动量 $P$ 是多少？用 $R, \theta, Z, u$ 及其时间导数表示答案。

(d) [0.5] What is the total angular momentum, $L$, about the $z$ axis of the system? Express the answer in terms of $R, \theta, Z, u$ and their time derivative.

(d) [0.5] 关于系统 z 轴的总角动量 $L$ 是多少？用 $R, \theta, Z, u$ 及其时间导数表示答案。

From now on, we introduce the dynamical variables $q(t) = u - Z$ and the center of mass coordinate of the system $Q(t) = \frac{mu+2MZ}{m+2M}$.

从现在开始，我们引入动力学变量$q(t) = u - Z$ 和系统的质心坐标 $Q(t) = \frac{mu+2MZ}{m+2M}$.。

(e) [2] Find the equation of motion for $q(t)$. Express your answer in terms of $\ddot{q}, q, R$ and given physical parameters.

(e) [2] 找出 $q(t)$ 的运动方程。用 $\ddot{q}, q, R$ 和给定的物理参数表达你的答案。

(f) [2] Find the equation of motion for $R(t)$. Express your answer in terms of $\ddot{R}, R, q, L$ and given physics parameters.

(f) [2] 求出 $R(t)$ 的运动方程。用 $\ddot{R}, R, q, L$ 和给定的物理参数表达你的答案。

(g) [2] In the limit of small planetary mass $m \ll M$ we can ignore the effect of the planet on the motion of the suns. Find the explicit solution for the motion of the suns $R(t)$ for orbits with small eccentricity $\epsilon \ll 1$. Write your solution as circular motion plus a term proportional to $e$, i.e. $R(t) = R_0 + \epsilon R_1(t)$ where $R_0$ is the radius of the circular orbit. You can assume the initial condition $R(0) = R_0(1+\epsilon)$.

(g) [2] 在小行星质量 $m \ll M$ 的极限下，我们可以忽略行星对太阳运动的影响。对于小偏心率 $\epsilon \ll 1$ 的轨道，找到太阳运动 $R(t)$ 的显式解。将你的答案写成圆周运动加上与 $\epsilon$ 成比例的项，即 $R(t) = R_0 + \epsilon R_1(t)$，其中 $R_0$ 是圆形轨道的半径。你可以假设初始条件 $R(0) = R_0(1+\epsilon)$。

(h) [2] Obtain the equation of motion for $q(t)$ in the limit $m \ll M$. You can see that $q(t)$ is a nonlinear oscillator driven by a nonlinear force term.

(h) [2] 求 $q(t)$ 在极限$m \ll M$时的运动方程。你可以看到 $q(t)$ 是一个由非线性力项驱动的非线性振荡器。

Solution:

(a) The total kinetic energy is

$$T = M\dot{R}^2 + M\dot{Z}^2 + MR^2\dot{\theta}^2 + \frac{1}{2}m\dot{u}^2$$

(b) The total potential energy is

$$V = -\frac{GM^2}{2R} - \frac{2GMm}{\sqrt{R^2 + (u-Z)^2}}$$

(c) The total linear momentum is

$$P = m\dot{u} + 2M\dot{Z}$$

(d) The total angular momentum is

$$L = 2MR^2\dot{\theta}$$

(e) We can rewrite $u$ and $Z$ in term of $q$ and $Q$,

$$u = Q + \frac{2M}{m+2M}q$$

$$Z = Q - \frac{m}{m+2M}q$$

The EOM for the planet is

$$m\ddot{u} = -\frac{2GMm}{(R^2+q^2)^{3/2}}q$$

Since

$$u = Q + \frac{2M}{m+2M}q \Rightarrow \ddot{u} = \frac{2M}{m+2M}\ddot{q}$$

$$\Rightarrow \frac{2mM}{m+2M}\ddot{q} = -\frac{2GMm}{(R^2+q^2)^{3/2}}q$$

$$\Rightarrow \mu\ddot{q} = -\frac{2GMm}{(R^2+q^2)^{3/2}}q$$

where the reduced mass of the system $\mu$ is defined by:

$$\frac{1}{\mu}=\frac{1}{2M}+\frac{1}{m}$$

(f) The total energy can be rewritten as,

$$E=T+V=M\dot{R}^2+M\dot{Z}^2+MR^2\dot{\theta}^2+\frac{1}{2}m\dot{u}^2-\frac{GM^2}{2R}-\frac{2GMm}{\sqrt{R^2+(u-Z)^2}}$$

$$E=M\dot{R}^2+M\left(\dot{Q}-\frac{m}{m+2M}\dot{q}\right)^2+MR^2\dot{\theta}^2+\frac{1}{2}m\left(\dot{Q}+\frac{m}{m+2M}\dot{q}\right)^2-\frac{GM^2}{2R}-\frac{2GMm}{\sqrt{R^2+q^2}}$$

$$E=M\dot{R}^2+\frac{1}{2}(m+2M)\dot{Q}^2+\frac{Mm}{m+2M}\dot{q}^2+MR^2\dot{\theta}^2-\frac{GM^2}{2R}-\frac{2GMm}{\sqrt{R^2+q^2}}$$

$$E=M\dot{R}^2+\frac{1}{2}(m+2M)\dot{Q}^2+\frac{1}{2}\mu\dot{q}^2+MR^2\dot{\theta}^2-\frac{GM^2}{2R}-\frac{2GMm}{\sqrt{R^2+q^2}}$$

$$\Rightarrow E=M\dot{R}^2+\frac{P^2}{2(m+2M)}+\frac{1}{2}\mu\dot{q}^2+MR^2\left(\frac{L}{2MR^2}\right)^2-\frac{GM^2}{2R}-\frac{2GMm}{\sqrt{R^2+q^2}}$$

$$\Rightarrow E=\frac{P^2}{2(m+2M)}+M\dot{R}^2-\frac{GM^2}{2R}+\frac{L^2}{4MR^2}+\frac{1}{2}\mu\dot{q}^2-\frac{2GMm}{\sqrt{R^2+q^2}}$$

where the momentum and angular momentum

$$p=m\dot{u}+2M\dot{Z}=m\left(\dot{Q}+\frac{2M}{m+2M}\dot{q}\right)+2M\left(\dot{Q}-\frac{2M}{m+2M}\dot{q}\right)=(m+2M)\dot{Q}$$

$$L=2MR^2\dot{\theta}$$

are conserved.

The constant energy gives

$$\frac{dE}{dt}=0\Rightarrow 2M\dot{R}\ddot{R}+\frac{GM^2}{2R^2}\dot{R}-\frac{L^2}{2MR^3}\dot{R}+\mu\dot{q}\ddot{q}+\frac{GMm}{(R^2+q^2)^{\frac{3}{2}}}\left(2R\dot{R}++2q\dot{q}\right)=0$$

$$\Rightarrow\left(2M\ddot{R}-\frac{L^2}{2MR^3}+\frac{GM^2}{2R^2}+\frac{2GMmR}{(R^2+q^2)^{\frac{3}{2}}}\right)\dot{R}+\left(\mu\ddot{q}+\frac{2GMmq}{(R^2+q^2)^{\frac{3}{2}}}\right)\dot{q}=0$$

$$\Rightarrow 2M\ddot{R}-\frac{L^2}{2MR^3}+\frac{GM^2}{2R^2}+\frac{2GMmR}{(R^2+q^2)^{\frac{3}{2}}}=0$$

(g) In the limit $\frac{m}{M}\to 0$,

$$2M\ddot{R}-\frac{L^2}{2MR^3}+\frac{GM^2}{2R^2}+\frac{2GMmR}{(R^2+q^2)^{\frac{3}{2}}}=0$$

$$\Rightarrow \frac{2}{M}\ddot{R} - \frac{L^2}{2M^3R^3} + \frac{G}{2R^2} + \frac{2GR}{(R^2+q^2)^{\frac{3}{2}}}\frac{m}{M} \approx \frac{2}{M}\ddot{R} - \frac{L^2}{2M^3R^3} + \frac{G}{2R^2} = 0$$

The orbit, $R(t) = R_0 + \epsilon R_1(t)$,

$$\frac{2\epsilon}{M}\ddot{R}_1 - \frac{L^2}{2M^3R_0^3}\left(1 - \frac{3\epsilon R_1}{R_0}\right) + \frac{G}{2R_0^2}\left(1 - \frac{2\epsilon R_1}{R_0}\right) = 0$$

$$\Rightarrow \frac{L^2}{2M^3R_0^3} = \frac{G}{2R_0^2} \Rightarrow R_0 = \frac{L^2}{GM^3}$$

And

$$\frac{2\epsilon}{M}\ddot{R}_1 = -\frac{L^2}{2M^3R_0^3}\frac{3R_1}{R_0} + \frac{G}{2R_0^2}\frac{2\epsilon R_1}{R_0} = -\left(\frac{G\epsilon}{2R_0^3}\right)R_1$$

$$\Rightarrow \ddot{R}_1 = -\frac{GM}{4R_0^3}R_1$$

$$\Rightarrow R_1 = A\cos\left(\sqrt{\frac{GM}{4R_0^3}}t + \phi\right)$$

The radial coordinate of the suns are

$$R_1(t) = R_0\left(1 + A\epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t + \phi\right)\right) = R_0\left(1 + \epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)\right)$$

(h) In the limit $m \ll M$, $\mu \approx m$ and

$$(R^2+q^2)^{-\frac{3}{2}} = \left(R_0^2\left(1 + \epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)\right)^2 + q^2\right)^{-\frac{3}{2}} \approx (R_0^2+q^2)^{-\frac{3}{2}}\left(1 + \frac{2R_0^2}{R_0^2+q^2}\epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)\right)^{-\frac{3}{2}}$$

$$\approx \frac{1}{(R_0^2+q^2)^{\frac{3}{2}}} - \frac{3R_0^2}{(R_0^2+q^2)^{5/2}}\epsilon\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right)$$

Finally, we obtain the EOM for $q(t)$,

$$m\ddot{q} + \frac{2GMm}{(R_0^2+q^2)^{\frac{3}{2}}}q - \frac{6GMmR_0^2\epsilon}{\left((R_0^2+q^2)^{\frac{5}{2}}\right)}\cos\left(\sqrt{\frac{GM}{4R_0^3}}t\right) = 0$$

The dynamics of this system given by the EOM for $q(t)$ turns out to be complex and chaotic even in the limit $m \ll M$. If you want to learn more about the complex dynamic (chaos) of this system, you can search the "Slitnikov Problem" in the internet.

3. [10 points] A massless rod can rotate without friction about the pivot point at its center. Light, propagating as a plane wave, propagates from left to right, along the $x$ axis. The electric field of the light is given by

3. [10 分] 一根无质量的杆可以绕其中心的枢轴点无摩擦地旋转。 平面光波沿 $x$ 轴从左向右传播。 光的电场由下式给出

$$\vec{E}(x,t) = E_0 \hat{y} \cos(kx - \omega t)$$

where $\vec{k} = k\hat{x}$ and $E_0$ is a real number. The angle between the rod and $\hat{y}$ is denoted by $\theta$.
其中 $\vec{k} = k\hat{x}$ 且 $E_0$ 为实数。杆和 $\hat{y}$ 之间的角度用 $\theta$ 表示。

Cantered at the ends of the rod are disks, each with one side perfectly mirror with 100% reflection and the other side with 100% absorbing. The disks are oriented so that light in the upper part of the rob (above the pivot) always strikes an absorptive surface, while in the lower part, it strikes a reflective surface. Each of the disks have mass $m$ and radius $r$. Assume that the distance $R$ from the pivot to the center of each disk satisfies $R \gg r$.
在杆的两端是圆盘，每个圆盘的一侧是 100% 反射的完美镜面，另一侧 100% 吸收。圆盘的方向使得杆上部（枢轴上方）的光总是照射到吸收面，而在下部，它照射到反射面。每个圆盘的质量为 $m$，半径为 $r$。假设枢轴点到每个圆盘中心的距离$R$ 满足$R \gg r$。

The Poynting vector, which describes the energy flux density (i.e. the energy per unit area per unit time) is given by
描述能量通量密度（即每单位时间每单位面积的能量）的坡印亭矢量由下式给出

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}),$$

where $\mu_0$ is the vacuum permeability. The momentum density $\vec{p}$ carried by the EM wave is
其中 $\mu_0$ 是真空磁导率。 光波携带的动量密度 $\vec{p}$ 为

$$\vec{p} = \frac{1}{c^2}\vec{S},$$

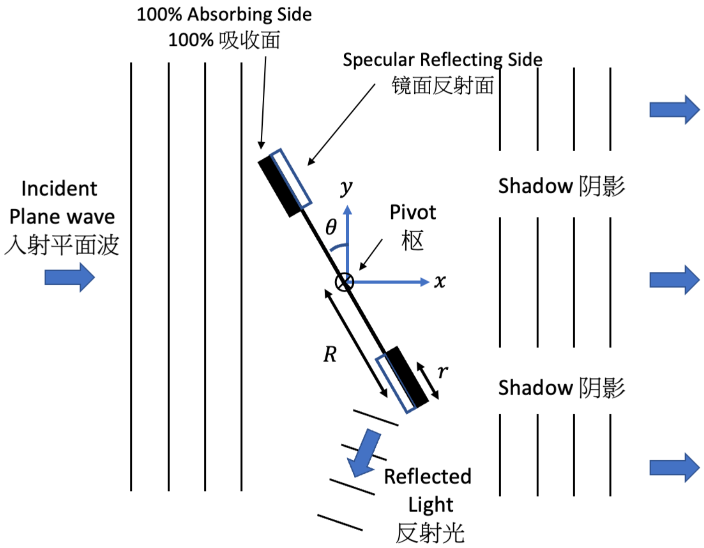
where $c$ is the speed of light in vacuum.
其中 $c$ 是真空中的光速。

(a) [1] What are the frequency $f$, wavelength $\lambda$, and magnetic field $\vec{B}(x,t)$ of the light?
(a) [1] 光的频率 $f$、波长 $\lambda$ 和磁场 $\vec{B}(x,t)$ 分别是什么？

(b) [1] What is the time-averaged Poynting vector of the incident light?
(b) [1] 入射光的时间平均坡印廷矢量是什么？

(c) [6] What is the total torque which is delivered by the light to the system of rod plus disks around the pivot point at a given angle $\theta$? What is the average torque over a full rotation of the rod?
(c) [6] 在给定角度 $\theta$ 下，光传递到围绕枢轴点的杆加圆盘系统的总扭矩是多少？杆旋转一圈的平均扭矩是多少？

(d) [2] Find the average angular acceleration of the rod over a revolution.
(d) [2] 求杆旋转一圈的平均角加速度。




Solution:

(a) $f = \frac{\omega}{2\pi}$, $\lambda = \frac{c}{f} = \frac{2\pi}{k}$ and $\vec{B}(x,t) = \frac{E_0}{c}\hat{z}\, cos\,(kx - \omega t)$

(b) $\vec{E}$ and $\vec{B}$ are perpendicular and in phase. The time average of $\cos^2 \omega t$ is $\frac{1}{2}$, so that

$$\langle \vec{S} \rangle = \frac{E_0^2}{2c\mu_0}$$

(c) The force exerted on the disk in the direction normal to its surface is given by the total momentum per second transferred by the light in this direction. The momentum transferred per unit area per second is $p_n c \cos\theta = pc\cos^2\theta$, where $p_n$ is the component of momentum density of the wave along the normal to the disk surface. The total momentum transfer per second to the absorbing disk (i.e. the force) is then

$$F_{abs} = (pc\cos^2\theta)(\pi r^2) = \frac{S}{c}\pi r^2 \cos^2\theta = \frac{1}{2c^2\mu_0}E_0^2 \pi r^2 \cos^2\theta$$

For the reflecting surface, the corresponding total momentum transfer is twice as much,

$$F_{ref} = \frac{1}{c^2\mu_0}E_0^2 \pi r^2 \cos^2\theta$$

(d) The torque is given by the sum of $\vec{r} \times \vec{F}$. The net torque is

$$\vec{\tau} = \frac{1}{2c^2\mu_0}E_0^2 \pi r^2 R \cos^2\theta\, \hat{z}$$

Taking time average over one full rotation in $\theta$,

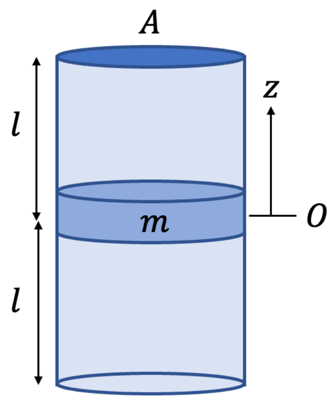
$$\langle \vec{\tau} \rangle = \frac{1}{4c^2\mu_0}E_0^2 \pi r^2 R \hat{z}$$

(e) Since $\tau = I\alpha$ where $I$ is the moment of inertia of the system about the pivot. In the limit $R \gg r$, we have $I = 2mR^2$ and

$$\langle \ddot{\theta} \rangle = \frac{E_0^2 \pi r^2}{8mRc^2\mu_0}$$

**4. [10 points]** A vertical, insulated and sealed cylinder with a cross-sectional area $A$, and an insulating piston of mass $m$ inside, whose thickness is negligible compared with the length of the cylinder. At the beginning, the piston is fixed in the center of the cylinder, which divides the cylinder into two air chambers with the same length $l$, as shown in the figure. Assume that the upper and lower gas chambers of the cylinder each contain $n$ moles of monatomic ideal gas with temperature $T_0$. In the following problems, it can be assumed that there is little friction between the piston and the cylinder wall, and that $l$ is much larger than the distance traveled by the piston ($\gg z$).

4. [10 分] 有一垂直豎立的密閉絕熱圓筒，截面積為 A，內裝有一質量為$m$的絕熱活塞，其厚度和圓筒的長度相比，可忽略不計。起始時，活塞被固定在圓筒的中央，將圓筒分隔成兩個長度同為 $l$ 的氣室，如圖所示。設圓筒的上下兩氣室各含有溫度為$T_0$ 和$n$ 摩耳的單原子分子理想氣體。在下列的問題中，可以假定活塞與圓筒壁之間的摩擦力很少，且 $l$ 遠大於活塞所移動的距離 ($l \gg z$)。




(a) [4] Release the piston from rest at time $t = 0$ so that it can move freely up and down. Find the trajectory of the piston $z(t)$. You can neglect the friction between the piston and the cylinder in the part.

(a) [4] 在時間 $t = 0$時從靜止中釋放活塞，使其能自由上下運動， 找出活塞的軌跡 $z(t)$。在这部分，你可以忽略活塞和气缸之间的摩擦。

(b) [2] How does the temperature of the upper and lower chambers in the cylinder change with the position of the piston, $z$?

(b) [2] 圓筒內上下兩氣室的溫度如何隨活塞位置 $z$ 的變動而改變?

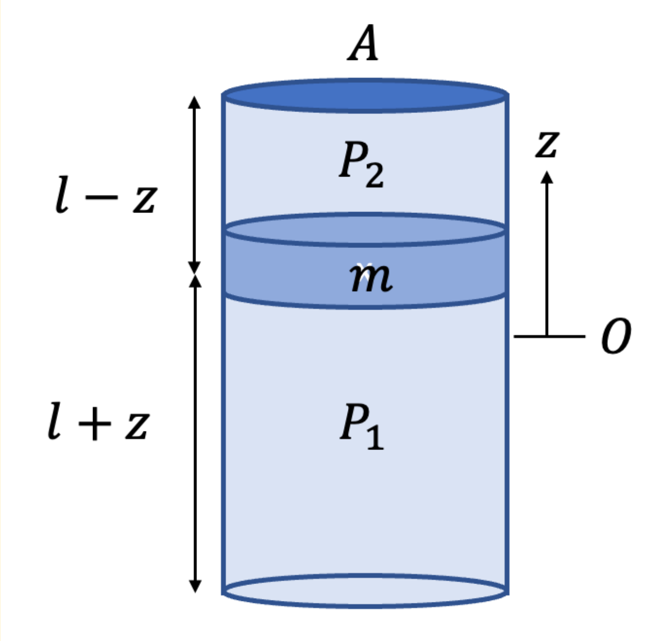
(c) [4] Although there is little friction between the piston and the cylinder, the piston will eventually come to rest after a long time. Find the position of the piston, $z_f$, when it rests and the temperature, $T_f$, of the gas in the cylinder at that time. . We can assume that the heat capacity of the cylinder and the piston is negligible, the temperature of the upper and lower chambers will eventually come to the same because of the movement of the piston and all heat lost due to friction will transfer into the internal energy of the gas.

(c) [4] 活塞與圓筒壁間的摩擦力雖然很少，但經過一段長時間後，終會使活塞靜止下來。試求活塞最後靜止時的位置 $z_f$ 和其時筒內氣體的溫度 $T_f$？我們假設圓筒壁和活塞的熱容量可忽略不計，且由於活塞的運動，使上下氣室溫度最後趨於一致，而且由于摩擦而损失的所有热量都将转化为气体的内能。

1

**Solution:**

(a) The process is adiabatic and we have $PV^\gamma = C$ where $\gamma = \frac{C_p}{C_V} = \frac{5}{3}$.




From the figure, we have

$$P_1 V_1^\gamma = P_2 V_2^\gamma = P_0 V_0^\gamma$$

$$\Rightarrow P_1 (l+z)^\gamma = P_2 (l-z)^\gamma = P_0 l^\gamma$$

$$P_1 = P_0 \left( \frac{1}{1 + \frac{z}{l}} \right)^\gamma \approx P_0 \left(1 - \frac{\gamma z}{l}\right)$$

$$P_2 \approx P_0 \left(1 + \frac{\gamma z}{l}\right) = P_0 \left(1 + \frac{5z}{3l}\right)$$

Newton's 2nd law gives

$$-P_0 A \left(1 + \frac{\gamma z}{l}\right) + P_0 A \left(1 - \frac{\gamma z}{l}\right) - mg = -\frac{2P_0 A \gamma}{l} z - mg = m\ddot{z}$$

$$\Rightarrow \ddot{z} = -\frac{2P_0 A \gamma}{ml} z - g$$

$$\Rightarrow \ddot{z} = -\omega^2 \left(z + \frac{g}{\omega^2}\right)$$

where $\omega = \sqrt{\frac{2P_0 A\gamma}{ml}}$. The general solution of $z(t)$ is:

$$z(t) = A\cos(\omega t + \phi) - \frac{g}{\omega^2}$$

From the initial conditions $z(0) = \dot{z}(0) = 0$,

$$\Rightarrow A\cos\phi - \frac{g}{\omega^2} = 0$$

$$-A\omega\sin\phi = 0$$

$$\Rightarrow z(t) = -\frac{g}{\omega^2}(1-\cos\omega t)$$

(b)

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} = T_0V_0^{\gamma-1}$$

$$\Rightarrow T_1 = T_0\left(\frac{1}{1+\frac{z}{l}}\right)^{\gamma-1} \approx T_0\left(1-\frac{(\gamma-1)z}{l}\right)$$

$$T_2 = T_0\left(1+\frac{(\gamma-1)z}{l}\right) = T_0\left(1+\frac{2z}{3l}\right)$$

(c) At equilibrium $z_0$, both chambers have the same temperature $T$ and the pressure satisfies

$$(P_1 - P_2)A = mg$$

$$\Rightarrow \left(\frac{nRT}{A(l+z_0)} - \frac{nRT}{A(l-z_0)}\right)A = mg$$

$$\Rightarrow \frac{1}{l+z_0} - \frac{1}{l-z_0} = \frac{mg}{nRT} \quad (1)$$

By energy conservation, we have

$$2\times\left(\frac{3}{2}nRT_0\right) = 2\times\left(\frac{3}{2}nRT\right) + mgz_0$$

$$\Rightarrow T = T_0 - \frac{mg}{3nR}z_0 \quad (2)$$

From equations (1) and (2), we can eliminate $T$ and get

$$\frac{1}{l+z_0} - \frac{1}{l-z_0} = -\frac{2z_0}{l^2-z_0^2} = \frac{mg}{nRT} = \frac{mg}{nR\left(T_0 - \frac{mg}{3nR}z_0\right)}$$

$$\Rightarrow l^2 - z_0^2 = -\frac{2nR}{mg}z_0\left(T_0 - \frac{mg}{3nR}z_0\right)$$

$$\Rightarrow z_0^2 - \frac{6nRT_0}{5mg}z_0 + \frac{3}{5}l^2 = 0$$

$$\Rightarrow z_0 = \left(\frac{3nRT_0}{5mg}\right) \pm \sqrt{\left(\frac{3nRT_0}{5mg}\right)^2 + \frac{3l^2}{5}}$$

Since $z_0 < 0$, we get

$$z_0 = \left(\frac{3nRT_0}{5mg}\right) - \sqrt{\left(\frac{3nRT_0}{5mg}\right)^2 + \frac{3l^2}{5}}$$

Note: In the limit when $\left(\frac{3nRT_0}{5mg}\right)^2 \gg \frac{3l^2}{5}$ $\left(\text{i.e. } \frac{nRT_0}{mg} \gg l\right)$, we have

$$z_0 = \left(\frac{3nRT_0}{5mg}\right)\left(1 - \sqrt{1 + \frac{3l^2}{5}\left(\frac{5mg}{3nRT_0}\right)^2}\right) \approx -\frac{1}{2}\left(\frac{3nRT_0}{5mg}\right)\frac{3l^2}{5}\left(\frac{5mg}{3nRT_0}\right)^2 = -\frac{3l^2}{10}\frac{5mg}{3nRT_0} = -\frac{mgl^2}{2nRT_0}$$

This result is justified if

$$|z_0| = \frac{mgl^2}{2nRT_0} \ll l \Rightarrow l \ll \frac{2nRT_0}{mg} \sim \frac{nRT_0}{mg}$$

is valid at high temperature $T_0 \gg \frac{mgl}{nR}$ and $z \ll l$.

And the temperature is

$$T = \frac{4}{5}T_0 + \sqrt{\left(\frac{T_0}{5}\right)^2 + \frac{1}{15}\left(\frac{mgl}{nR}\right)^2} = T_0\left(\frac{4}{5} + \frac{1}{5}\sqrt{1 + \frac{5}{3T_0^2}\left(\frac{mgl}{nR}\right)^2}\right) \approx T_0\left(1 + \frac{m^2g^2l^2}{6n^2R^2T_0^2}\right)$$

**~ End of Part 1 卷-1 完 ~**