Pan Pearl River Delta Physics Olympiad 2023
2023 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助

Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 29 January 2023)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Vacuum bubbles (28 points)
问题 1: 真空泡泡 (28 分)

Is our vacuum stable? We don't know. It's possible that we do not live in the true vacuum. Rather, we live in a false vacuum which can decay into true vacuum by emerging and expanding bubbles. To describe such a possibility, we will make use of a space-time dependent "scalar field" $\phi(t,x,y,z)$, which takes a real value at every space-time point. (Similar to height on a map, which takes a real value at every point on the x-y plane, while a scalar field takes a real value for any given t,x,y,z. Also, in a full quantum theory, we have to distinguish operators and numbers, but here we will assume the scalar field only take real number values.)

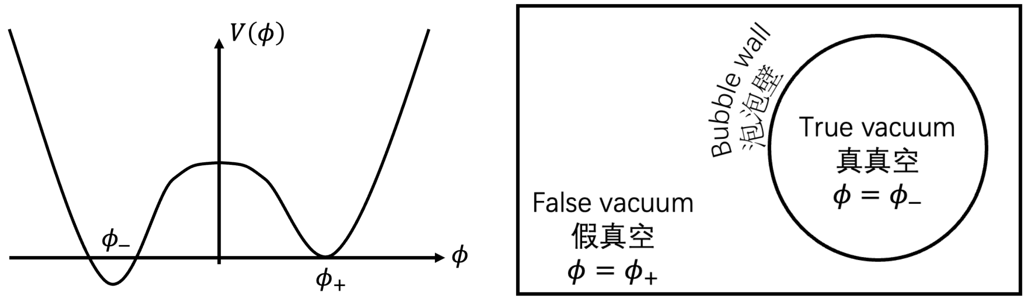
我们的真空是稳定的吗？我们不知道。有可能我们并不是生活在一个稳定的真真空（既真的“真空”）里面。我们可能生活在一个假真空里，而假真空可以通过自发产生和膨胀的泡泡衰变到真真空。为了描述这种可能，我们将使用一个依赖于时空点到“标量场”$\phi(t,x,y,z)$：它在每个时空点上取一个实数。（这有点像地图上的高度，在 x-y 平面的每个点上取一个实数。不过标量场是在每个t,x,y,z时空点上取一个实数。另外，在完整的量子理论中，我们需要考虑算符和数的区别，但这里我们假设标量场的取值仅是普通的实数。）

The scalar field satisfies the following equation of motion: $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}+\frac{dV(\phi)}{d\phi}=0$, where $V(\phi)$ is the potential density of the field, and we will call it potential for short in this problem. The energy density of the scalar field is $\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)$.

标量场满足运动方程 $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}+\frac{dV(\phi)}{d\phi}=0$，其中$V(\phi)$是场的势能密度，我们将简称它为标量场的势能。标量场的能量密度是 $\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)$。

We consider the following potential: the false vacuum has field value $\phi=\phi_+$ where $V(\phi_+)=0$, and the true vacuum has field value $\phi=\phi_-$, where $V(\phi_-)$ is slightly negative. In the left panel of the following figure, we plot the shape of the potential. The right panel is an example of the false and true vacuum in position space.

我们考虑如下势能：假真空处标量场取值是 $\phi=\phi_+$，满足 $V(\phi_+)=0$；真真空处标量场取值是$\phi=\phi_-$， $V(\phi_-)$ 取一个接近 0 的负值。下图（左）是势能的函数形式，下图（右）是位置空间中假真空和真真空的一个例子。



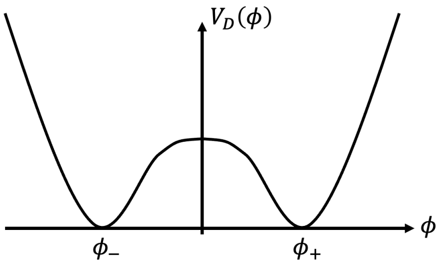
In this problem, we will use natural units and set the speed of light in vacuum $c=1$ (by redefining the time unit as the time that light traveled over unit length). In this unit, when an object is at rest, its energy equals its mass by the famous formula $E=mc^2=m$.

本题中，我们将使用自然单位并定义真空中的光速 $c=1$（既定义时间的单位为光穿越单位长度的时间）。在自然单位制中，物体的静止能量等于它的质量，这就是著名的公式 $E=mc^2=m$。

A. DOMAIN WALL 畴壁

Before coming to the asymmetric potential which generates vacuum bubbles, let us consider a symmetric potential as follows:

在我们研究非对称的势能以及它生成的真空泡泡前，我们先考虑如下一个对称的势能：



Let's find a static solution which is homogeneous along the y and z directions, known as a domain wall. The potential of the domain wall $V(\phi) = V_D(\phi)$ is illustrated above, with two minima $V_D(\phi_\pm) = 0$. The domain wall can be used as the local approximation of the bubble wall.

我们将找一种在 y 和 z 方向均匀的 “畴壁” 解。上图是畴壁解对应的势能 $V(\phi) = V_D(\phi)$，它具有两个最小值 $V_D(\phi_\pm) = 0$。畴壁可以作为泡泡壁的局域近似。

A1. SIMPLIFY THE EQUATION OF MOTION 化简运动方程

A1	Given the static and homogeneity (independency of y and z) conditions, write down the simplified equation of motion for ϕ. 根据静态，以及均匀（不依赖于y 和 z 方向）的条件，写出ϕ化简后的运动方程。	2 points 2 分

Solution: $\frac{d^2\phi}{dx^2} = \frac{dV_D(\phi)}{d\phi}$. [Note, if the student did not convert ∂ to d, we also consider it as correct. Also, since in Part A, $V = V_D$, both are correct.]

A2. THE DOMAIN WALL PROFILE 畴壁上标量场取值的空间变化

A2	Express $\frac{d\phi}{dx}$ in terms of $V_D(\phi)$. 请把 $\frac{d\phi}{dx}$ 用 $V_D(\phi)$ 表达出来。	3 points 3 分

Solution: $\frac{d^2\phi}{dx^2} = \frac{dV_D}{d\phi} = \frac{dV_D}{dx}\frac{dx}{d\phi}$, thus $\frac{1}{2}\frac{d}{dx}\left(\left(\frac{d\phi}{dx}\right)^2\right) = \frac{dV_D}{dx}$. Since far away from the domain wall, we have boundary condition $\frac{d\phi}{dx} = 0$ and $V_D = 0$, we have $\left(\frac{d\phi}{dx}\right)^2 = 2V_D$.

A3. THE DOMAIN WALL TENSION 畴壁的张力 (2')

A3	The tension of the domain wall (energy density of the wall for unit area in the y and z directions) is $\sigma = \int_{\phi_-}^{\phi_+} f(\phi)d\phi$. Find $f(\phi)$ in terms of $V_D(\phi)$. 畴壁的张力（既在 y、z 方向单位面积上，畴壁的能量密度）是 $\sigma = \int_{\phi_-}^{\phi_+} f(\phi)d\phi$。请把 $f(\phi)$用 $V_D(\phi)$ 表达出来。	2 points 2 分

Note: to avoid propagation of possible errors, in the later part of the questions, please still use the domain wall tension σ where applicable, instead of using the integral expression that you obtain.

注：为避免潜在的错误传播，在本题后面的部分中，当用到畴壁张力时，请仍使用符号 σ ，而不是这里你求出的积分表达式。

Solution: $\sigma = \int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V_D(\phi)\right]dx = \int_{\phi_-}^{\phi_+}[2V_D(\phi)]\frac{dx}{d\phi}d\phi = \int_{\phi_-}^{\phi_+}[2V_D(\phi)]^{1/2}d\phi$

$$f(\phi) = \sqrt{2V_D(\phi)}$$

B. BUBBLE WALL 泡泡壁

If we only look at a small part, a bubble wall can be approximated as a domain wall. But globally, the bubble can be approximated to be spherical with radius R. Let's assume that R is large enough, such that the thickness of the bubble wall is much smaller than R (thin wall approximation). Inside and outside the bubble, $\phi \to \phi_\pm$ exponentially quickly.
如果我们只看一泡泡壁的一小部分的话，泡泡壁上的一小块可以用畴壁来近似。但是整体上，真空泡泡可以近似为球形的，具有半径 R。假设 R 足够大，泡泡壁的厚度远远小于 R（薄壁近似）。在泡泡壁的内部和泡泡壁的外部，ϕ 指数快地趋向于 $\phi_\pm$。

At the moment when a bubble is nucleated, the bubble is static, and the bubble nucleation and motion create negligible amount of radiation or other dissipations.
在真空泡泡产生的时刻，泡泡是静止的。泡泡产生的过程带来的辐射或其它耗散可以忽略。

B1. THE ENERGY ON THE BUBBLE WALL 泡泡壁的能量 (1')

B1	At the momentum of bubble nucleation, calculate the energy E_W carried by the bubble wall using R and the bubble tension σ. 在真空泡泡产生的时刻，利用 R 和泡泡壁的张力 σ 计算泡泡壁的能量 E_W。	1 point 1 分

Solution: $E_W = 4\pi R^2 \sigma$.

B2. FALSE AND TRUE VACUA 假真空和真真空

B2	For a spherical bubble to appear, there must be an energy density difference between $V(\phi_\pm)$. Thus, to write down a potential to model bubble nucleation, we consider the potential $V(\phi) = V_D(\phi) + \frac{\epsilon}{\phi_+ - \phi_-}(\phi - \phi_+)$. In the thin wall approximation, we are only interested in leading order results in ϵ (the lowest order in Taylor expansion which contains ϵ). Calculate ϵ using σ and R. 为了让球形真空泡泡能够出现，$V(\phi_\pm)$ 的取值必须不同。所以，为了对真空泡泡的产生过程建立势能模型，我们考虑势能 $V(\phi) = V_D(\phi) + \frac{\epsilon}{\phi_+ - \phi_-}(\phi - \phi_+)$。在薄壁近似下，我们只感兴趣 ϵ 零头阶效应（既泰勒展开中含有 ϵ 的最低阶）。利用 σ 和 R 计算 ϵ。	2 points 2 分

Solution:

The energy density inside the bubble is $-\epsilon$.

From energy conservation, $E_W = \frac{4}{3}\pi R^3 \epsilon$,

thus $\epsilon = \frac{3\sigma}{R}$.

Note: though the introduction of the linear term modifies the minima of the potential a little, but the modification multiplying the energy density will be $O(\epsilon^2)$ and thus neglected.

B3. BUBBLE MOTION 泡泡的运动

B3	At the moment of bubble nucleation, calculate the acceleration a of the bubble wall in terms of σ and R. 在真空泡泡产生的瞬间，利用 σ 和 R 计算泡泡的加速度 a。	2 points 2 分

Solution:

Consider bubble expansion $R \to R + \delta R$.

Since volume expands faster than surface, the energy obtained by the bubble wall from this expansion is $\delta E_W = \frac{4}{3}\pi(R+\delta R)^3\epsilon - 4\pi(R+\delta R)^2\sigma = 4\pi R\sigma\, \delta R$

Since the force $F = \frac{\delta E_W}{\delta R} = ma = E_W a$ (in the unit $c = 1$)

Acceleration $a = \frac{\delta E_W/\delta R}{E_W} = \frac{1}{R}$.

B4. BEYOND NEWTONIAN MECHANICS 超越牛顿力学

B4	When the speed of the bubble wall is close to the speed of light, Newtonian mechanics breaks down and special relativity should be used instead. In special relativity, the kinetic energy of a moving object is $E_K = (\gamma - 1)m$, where $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$.. Calculate the time needed from the nucleation of the bubble to that the speed of the bubble wall to reach 0.6. 当真空泡泡的速度接近光速，牛顿力学不再适用，我们应该使用狭义相对论。在狭义相对论里，运动物体的动能为 $E_K = (\gamma - 1)m$，其中 $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$。计算真空泡泡从产生到泡泡壁速度达到 0.6 所需要的时间。 Hint: you may need the mathematical relation $\frac{d\sqrt{x^2-1}}{dx} = \frac{x}{\sqrt{x^2-1}}$. 提示：你可能需要数学关系 $\frac{d\sqrt{x^2-1}}{dx} = \frac{x}{\sqrt{x^2-1}}$。	4 points 4 分

Solution:

The rest + kinetic energy of the bubble is $E_W = \gamma m = \frac{1}{\sqrt{1-v^2}} 4\pi r^2 \sigma$, where $r = r(t)$ is the time-dependent radius of the bubble (to be distinguished from the initial radius of the bubble R.

Thus, energy conservation at time t yields $\frac{1}{\sqrt{1-v^2}} 4\pi r^2 \sigma = \frac{4}{3}\pi r^3 \epsilon$,

i.e., $\frac{1}{\sqrt{1-v^2}} = \frac{r}{R}$, i.e., $\frac{dr}{dt} = \frac{\sqrt{r^2-R^2}}{r}$, i.e., $\frac{rdr}{\sqrt{r^2-R^2}} = d\sqrt{r^2 - R^2} = dt$.

Given the initial condition $r(t = 0) = R$, we have $r^2 - R^2 = t^2$.

(Note, after a complicated calculation (it's more complicated if you use relativistic force), you get a surprisingly simple result. This is not a coincidence. In fact, this is an analytical continuation of a Euclidean 4-sphere. We will see a little bit of this in Part C.)

$v = \frac{dr}{dt} = \frac{t}{\sqrt{R^2+t^2}} = 0.6$, $t = 0.75R$.

C. NUCLEATION RATE OF THE BUBBLE 泡泡的产生率

What's the probability for a bubble to appear? It can be shown that the nucleation rate Γ of the bubble, i.e., the probability for a bubble to appear in unit volume and during unit time, can be written as $\Gamma \simeq Ae^{-S_E/\hbar}$, where A and $\hbar$ are constants, and S_E is a "Euclidean action", which can be calculated with the following procedure:

真空泡泡产生的概率是多少？可以证明，泡泡的产生率 Γ，即单位时间单位体积，一个泡泡产生的概率，可以由 $\Gamma \simeq Ae^{-S_E/\hbar}$ 计算，其中 A 和 $\hbar$ 是常数，S_E 是一个 “欧氏作用量” ，由以下步骤计算：

(1) We "rotate" our physical time t to "Euclidean time" $\tau = it$ (where $i^2 = -1$).

(1) 将物理时间 t“旋转”到欧氏时间 $\tau = it$（其中$i^2 = -1$）。

(2) The real-time and imaginary time field configurations are related by $\phi(t = 0, x, y, z) = \phi(\tau = 0, x, y, z)$.

(2) 实数时间和虚数时间到场位形由 $\phi(t = 0, x, y, z) = \phi(\tau = 0, x, y, z)$ 联系起来。

(3) Given the above time boundary condition, find a 4-dimensional rotational symmetric solution of the Euclidean equation of motion $\frac{\partial^2\phi}{\partial\tau^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{dV(\phi)}{d\phi}=0$.

(3) 给定上述时间边界条件，找到欧氏运动方程 $\frac{\partial^2\phi}{\partial\tau^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{dV(\phi)}{d\phi}=0$ 的四维旋转对称的解。

(4) Insert the solution to the Euclidean action $S_E=\int dt\, d^3x\left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)\right]$ to find Γ.

(4) 将找到的解带入欧氏作用量 $S_E=\int dt\, d^3x\left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dy}\right)^2+\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2+V(\phi)\right]$ 来求出 Γ。

Let us do this calculation in this part. Note that a 4-dimensional unit ball with radius r has "volume" $\frac{\pi^2}{2}r^4$ and surface "area" $2\pi^2r^3$.

让我们在这一部分中做上述计算。注意，四维球的“体积”为 $\frac{\pi^2}{2}r^4$，球面面积为 $2\pi^2r^3$。

C1. THE EUCLIDEAN EQUATION OF MOTION 欧氏运动方程

Since we are to look for a 4-dimentional rotational symmetric solution of the Euclidean equation of motion, it is convenient to use $\rho=\sqrt{\tau^2+x^2+y^2+z^2}$ as the variable for equation solving. Assuming that $\phi=\phi(\rho)$ only depends on ρ (i.e., 4-dimensional rotational symmetric), the general Euclidean equation of motion can be written as $\frac{d^2\phi}{d\rho^2}+f(\rho)\frac{d\phi}{d\rho}-\frac{dV(\phi)}{d\phi}=0$.

为了寻找运动方程的四维转动不变解，使用 $\rho=\sqrt{\tau^2+x^2+y^2+z^2}$ 作为解方程的变量比较方便。假设$\phi=\phi(\rho)$只依赖于ρ（这就是四维转动不变的含义），一般的欧氏运动方程可以写成 $\frac{d^2\phi}{d\rho^2}+f(\rho)\frac{d\phi}{d\rho}-\frac{dV(\phi)}{d\phi}=0$。

C1	Find the expression of $f(\rho)$. 求 $f(\rho)$。 Hint: if the formal calculation needs too much calculus, you can consider an example: by tuning the form of $V(\phi)$, one can obtain a solution $\phi=\rho^2$. Then $f(\rho)$ can be solved by this example (and this form $f(\rho)$ will apply for all forms of $V(\phi)$, not limited to this special form of solution). 提示：如果进行普适的计算需要太多微积分，你可以考虑一个例子：通过调整 $V(\phi)$ 的形式，我们得到一个解 $\phi=\rho^2$。这时，$f(\rho)$ 可以从这个例子里解出来（之后这个 $f(\rho)$ 的形式对所有 $V(\phi)$ 都适用，不仅限于这个特殊解）。	2 points 2 分

Solution:

Since the above equation is a general one, we can design a potential, such that the solution of the Euclidean equation is $\phi=\rho^2=\tau^2+x^2+y^2+z^2$. We have

$$\frac{\partial^2\phi}{\partial\tau^2}=\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\phi}{\partial z^2}=2$$

On the other hand, $\frac{d^2\phi}{d\rho^2}=2$, $\frac{d\phi}{d\rho}=2\rho$ and

$$8=\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\frac{d^2\phi}{d\rho^2}+f(\rho)\frac{d\rho}{d\rho}=2+2\rho f(\rho)$$

Then $2\rho\times f(\rho)=6, f(\rho)=3/\rho$.

C2. THE EUCLIDEAN ACTION 欧氏作用量

C2	Write the Euclidean action S_E as an integral of ρ from $\rho=0$ to $\rho\to\infty$. 以对 ρ 的积分（从$\rho=0$ 积到 $\rho\to\infty$）的形式写出欧氏作用量 S_E。	2 points 2 分

Solution: For 4-dimensional space, we have $d\tau dx dy dz=2\pi^2\rho^3 d\rho$ and

$S_E = 2\pi^2 \int_0^\infty \rho^3 \left(\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V\right) d\rho.$

C3. QUALITATIVE INSPECTION 定性讨论 (4')

Before trying to solve the Euclidean equation of motion, let us first see how it behaves. If you are not familiar with this Euclidean equation of motion, you can consider the following analogy: consider ϕ as the position of a particle, and ρ as an effective time variable. In this case, answer the following questions (choose one from the options):

在解欧氏运动方程之前，我们先看看方程的性质。如果你不熟悉欧氏运动方程，你可以用如下类比来理解：把 ϕ 类比为一个粒子的位置，把 ρ 类比为适用于这个粒子的有效时间变量。在这种情况下，回答以下问题（单项选择题）：

C3-1	What's the nature of $f(\rho)\frac{d\phi}{d\rho}$? 以下哪项是 $f(\rho)\frac{d\phi}{d\rho}$ 的性质？ (A) friction (i.e. decelerate the particle) 阻力（即让粒子减速） (B) anti-friction (i.e. accelerate the particle) 推力（即让粒子加速） (C) friction for $\frac{d\phi}{d\rho} > 0$ and anti-friction otherwise 在 $\frac{d\phi}{d\rho} > 0$ 情况下是阻力，否则是推力 (D) friction for $\frac{d\phi}{d\rho} < 0$ and anti-friction otherwise 在 $\frac{d\phi}{d\rho} < 0$ 情况下是阻力，否则是推力	1 point 1 分

C3-2	What's the force that drives the motion of the "particle" ϕ? 驱动粒子 ϕ 运动的力是哪个？ (A) V (B) $-V$ (C) $dV/d\phi$ (D) $-dV/d\phi$	1 point 1 分

C3-3	Where is the "starting point" of ϕ at $\rho = 0$? (Here δ is extremely small but finite) 在 $\rho = 0$，ϕ 的“起始位置”在哪里？（其中 δ 是一个非常小但有限的数） (A) $\phi_- - \lvert O(\delta)\rvert$ (B) ϕ_- (C) $\phi_- + \lvert O(\delta)\rvert$ (D) $\phi_+ - \lvert O(\delta)\rvert$ (E) ϕ_+ (F) $\phi_+ + \lvert O(\delta)\rvert$	1 point 1 分

C3-4	Where is the "end point" of ϕ at $\rho \to \infty$? 在 $\rho \to \infty$，ϕ 的“最终位置”在哪里？ (A) $\phi_- - \lvert O(\delta)\rvert$ (B) ϕ_- (C) $\phi_- + \lvert O(\delta)\rvert$ (D) $\phi_+ - \lvert O(\delta)\rvert$ (E) ϕ_+ (F) $\phi_+ + \lvert O(\delta)\rvert$	1 point 1 分

Solution: 1. A; 2. C; 3. C; 4. E

Note: For (C3-3) the solution is C instead of B. One can see it from two ways: First, consider x=y=z=0. In this case, $\rho = 0$ denotes the bubble center after bubble nucleation. Then from continuity of ϕ, the field value at the bubble center should have a slightly (by an exponentially small margin) larger field value than ϕ_-. The second way is considering the effective particle and the friction force. For (C3-4), at infinity, consider $\tau \to 0$, then it corresponds to spatial infinity, thus $\phi = \phi_+$ (spatial infinity makes it exact).

C4. THE BUBBLE NUCLEATION RATE 泡泡产生率

C3-4	Express S_E in terms of R and σ. 用 R 和 σ 写出 S_E。	4 points 4 分

Solution:

For the integral $S_E = 2\pi^2 \int_0^\infty \rho^3 \left(\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V\right) d\rho$

Note that at $\rho \to \infty$, the contribution vanishes. Thus, the integral breaks up to

(1) The contribution of the domain wall (replacing x with ρ):

(surface area) x (tension) = $2\pi^2 R^3 \sigma$.

Note: you may notice that the dimension of the problem here is different from the problem above. But at the domain wall, ρ is a constant and all the above calculation of the domain wall tension carries over to here.

(2) The contribution of the vacuum energy inside the bubble.

(volume) x (vacuum energy density) $= -\frac{1}{2}\pi^2 R^4 \epsilon$

Summing the two terms up and inserting $\epsilon = \frac{3\sigma}{R}$, we get

$S_E = \frac{1}{2}\pi^2 R^3 \sigma$.

Reference: "Fate of the false vacuum: Semiclassical theory", by Sidney Coleman, Phys. Rev. D **15**, 2929 (1977).

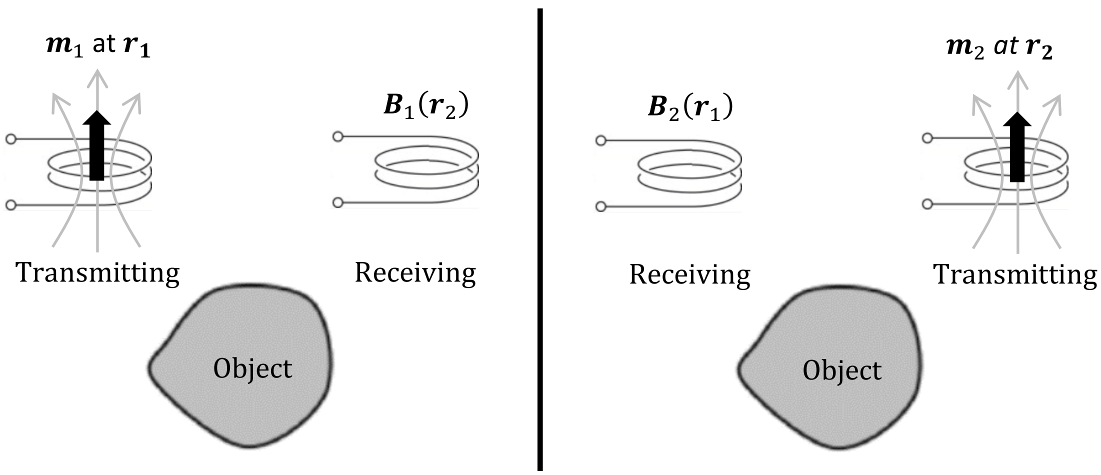
Problem 2: (32 points)
问题 1: (32 分)

Lorentz reciprocity is a fundamental principle in electromagnetism with important application in antenna design theory. It states that the receiving and transmitting capabilities of an antenna are identical. On the other hand, reciprocity can be broken by using magnetic materials under an external magnetic field with strong magneto-optical effect. The study of Lorentz reciprocity can be extended to nearly zero frequency at magnetostatics, shown in the figure below, with two current coils and an arbitrary object fixed in locations. When one current coil works in transmitting mode, another one works in receiving mode. If we model the two current coils as magnetic dipole moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ at locations $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, generating magnetic flux densities $\boldsymbol{B}_1(\boldsymbol{r})$ and $\boldsymbol{B}_2(\boldsymbol{r})$ in transmitting mode, respectively. Then, the ***reciprocity relationship*** can be expressed as

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$

洛伦兹互易性是电磁学的基本原理，在天线设计理论中具有重要应用。 它指出天线的接收和发射能力是相同的。 另一方面，在外磁场下，使用具有强磁光效应的磁性材料可以破坏互易性。 洛伦兹互易的研究可以扩展到静磁学，如下图所示，有两个电流环和一个固定在某个位置的物体。 当一个电流环在发射模式工作时，另一个电流环在接收模式工作。 如果我们将两个电流环设为位置 $\boldsymbol{r}_1$ 和 $\boldsymbol{r}_2$ 处的磁偶极子 $\boldsymbol{m}_1$ 和 $\boldsymbol{m}_2$，分别在发射模式下产生磁通密度 $\boldsymbol{B}_1(\boldsymbol{r})$ 和 $\boldsymbol{B}_2(\boldsymbol{r})$。 那么，***互易关系***可以表示为

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$



The magnetic field $\boldsymbol{B}(\boldsymbol{r})$ generated by a magnetic dipole $\boldsymbol{m}_i$ located at $\boldsymbol{r}_i$ is given by,

在位置 $\boldsymbol{r}_i$ 处的磁偶极子 $\boldsymbol{m}_i$ 所产生的磁场 $\boldsymbol{B}_i(\boldsymbol{r})$ 由下式给出，

$$\boldsymbol{B}_i(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{r}-\boldsymbol{r}_i)(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^5} - \frac{\boldsymbol{m}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right)$$

In this problem, we will first establish the reciprocity relationship and will investigate how it can be broken by imposing a constant velocity on the object.

本题中，我们将首先建立互易关系，并研究如何通过对物体施加恒定速度来打破它。

A. ESTABLISHING MAGNETOSTATIC RECIPROCITY AND NON-RECIPROCITY 建立静磁互易性和非互易性

For magnetostatics, we have vector potential $\boldsymbol{A}$, magnetic flux density $\boldsymbol{B}$, magnetic field strength $\boldsymbol{H}$ and impressed current density $\boldsymbol{J}$. These fields satisfy the Ampère's law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

with material response

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$

where $\mu(\boldsymbol{r})$ is the isotropic magnetic permeability profile for the material, i.e. the object in the figure.

对于静磁学，我们有矢量势 $\boldsymbol{A}$、磁通密度 $\boldsymbol{B}$、磁场强度 $\boldsymbol{H}$ 和外加电流密度 $\boldsymbol{J}$。这些场满足安培定律

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

关于物质反应的表达式为

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$

其中 $\mu(\boldsymbol{r})$ 是材料的各向同性磁导率分布，即图中的物体。

We want to establish the reciprocity relationship for magnetostatics when we have a magnetic dipole moment $\boldsymbol{m}_1$ at location $\boldsymbol{r}_1$ in one case and a magnetic dipole $\boldsymbol{m}_2$ at $\boldsymbol{r}_2$ in another case. The two dipoles generate magnetic fields $\boldsymbol{B}_1(\boldsymbol{r})$ and $\boldsymbol{B}_2(\boldsymbol{r})$, respectively. The two cases, labeled by $i = 1,2$, have the current density $\boldsymbol{J}_i = \nabla \times \boldsymbol{M}_i$ and magnetization $\boldsymbol{M}_i = \boldsymbol{m}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$ for the two dipoles.

在一种情况下$\boldsymbol{r}_1$处有一个磁偶极子 $\boldsymbol{m}_1$并产生磁场 $\boldsymbol{B}_1(\boldsymbol{r})$而在另一种情况下$\boldsymbol{r}_2$处有一个磁偶极子 $\boldsymbol{m}_2$并产生磁场 $\boldsymbol{B}_2(\boldsymbol{r})$。对于由 $i = 1,2$ 索引的两个偶极子，我们有电流密度 $\boldsymbol{J}_i = \nabla \times \boldsymbol{M}_i$和磁化强度 $\boldsymbol{M}_i = \boldsymbol{m}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$。以下我们想要建立静磁学的互易关系。

Here, we also give some formulas for these differential operators:

在这里，我们提供一些关于微分算子的公式：

$$\nabla \times \boldsymbol{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x)$$
$$\nabla \cdot \boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$$
$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{B} - \boldsymbol{A} \cdot \nabla \times \boldsymbol{B}$$
$$\int \nabla \cdot \boldsymbol{A} dV = \int \boldsymbol{A} \cdot d\boldsymbol{a}$$
$$\int \nabla \times \boldsymbol{A} dV = -\int \boldsymbol{A} \times d\boldsymbol{a}$$
$$\int \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} dV = \int \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} dV + \int \boldsymbol{A} \times \boldsymbol{B} \cdot d\boldsymbol{a}$$

and Kronecker delta function $\delta(\boldsymbol{r})$ is defined by

$$\delta(\boldsymbol{r}) = \begin{cases} \infty & \text{if } \boldsymbol{r} = \boldsymbol{0} \\ 0 & \text{otherwise} \end{cases}$$

which satisfies $\int \delta(\boldsymbol{r}) dV = 1$ when we integrate a volume V enclosing the origin. $\boldsymbol{a}$ is defined as the closed area enclosing volume V.

当我们对包含原点的一个体积 V 进行积分时，Kronecker delta 函数 $\delta(\boldsymbol{r})$ 定義為:

$$\delta(\boldsymbol{r}) = \begin{cases} \infty & \text{if } \boldsymbol{r} = \boldsymbol{0} \\ 0 & \text{除此以外} \end{cases}$$

而且满足 $\int \delta(\boldsymbol{r}) dV = 1$。 $\boldsymbol{a}$ 定义为包围体积 V 的封闭区域。

A1. PROVING THE RECIPROCITY RELATIONSHIP 证明互易关系

A1	Prove the reciprocity relationship $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$. Hint: you may consider $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2)$. 证明互易关系 $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$。 提示：你可以考虑 $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2)$。	3 points 3 分

Solution:

In magnetostatics, we have

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} = \mu \boldsymbol{H}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

where the free currents are from impressed current coils which can be described as magnetization $\boldsymbol{M}$ through

$$\boldsymbol{J} = \nabla \times \boldsymbol{M}$$

For two different sets of fields and currents labeled by subscript 1 and 2 (with same medium μ), we have

$$\begin{aligned} &\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1) \\ &= \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{H}_1 \cdot \boldsymbol{B}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 + \boldsymbol{H}_2 \cdot \boldsymbol{B}_1 \\ &= \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 - \mu \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 + \mu \boldsymbol{H}_2 \cdot \boldsymbol{H}_1 \end{aligned}$$

$$\Rightarrow \int \nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1) dV = \int \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 dV - \int \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 dV$$

There are only close-loop currents, i.e. made of magnetic dipoles. For each magnetic dipole, we have

$$\boldsymbol{B}_{\mathrm{i}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{r} - \boldsymbol{r}_i)(\boldsymbol{r} - \boldsymbol{r}_i) \cdot \boldsymbol{m}_i}{|\boldsymbol{r} - \boldsymbol{r}_i|^5} - \frac{\boldsymbol{m}_i}{|\boldsymbol{r} - \boldsymbol{r}_i|^3}\right)$$

As integration volume grows to infinity, the integral $\int \nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1) dV$ (with Gauss's law) decays to zero by

$$|\boldsymbol{H}| \propto 1/r^3, \qquad |\boldsymbol{A}| \propto 1/r^2, \qquad |\boldsymbol{H} \times \boldsymbol{A}| \propto 1/r^5$$

Then

$$\int \boldsymbol{J}_1 \cdot \boldsymbol{A}_2 dV = \int \boldsymbol{J}_2 \cdot \boldsymbol{A}_1 dV$$

By substituting $\boldsymbol{J} = \nabla \times \boldsymbol{M}$, we have

$$\int \boldsymbol{M}_1 \cdot \boldsymbol{B}_2 dV - \int \boldsymbol{M}_2 \cdot \boldsymbol{B}_1 dV = -\int \boldsymbol{M}_1 \times \boldsymbol{A}_2 \cdot d\boldsymbol{a} + \int \boldsymbol{M}_2 \times \boldsymbol{A}_1 \cdot d\boldsymbol{a}$$

Localized sources imply the right hand side goes to zero. Finally, from two point dipoles $\boldsymbol{m}_1$ at $\boldsymbol{r}_1$ and $\boldsymbol{m}_2$ at $\boldsymbol{r}_2$ for the two magnetization $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, we arrive

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) = \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$$

If the material conducts electricity with an electrical conductivity σ, we have to add an additional term to the current density $\boldsymbol{J}$ due to the free current through

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma \boldsymbol{E}$$

in the Ampère's law stated previously. Suppose now we move the conductor by a constant velocity $\boldsymbol{v}$. There will be a Lorentz force on the free charges proportional to $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$. It further updates the additional term in the current density through

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

which may upset the reciprocity relationship.

如果材料以电导率 σ 导电，由于通过的自由电流，我们必须为在此前安培定律中的电流密度 $\boldsymbol{J}$ 添加一个附加项

$$\boldsymbol{J} \to \boldsymbol{J} + \sigma \boldsymbol{E}.$$

假设现在我们以恒定速度 $\boldsymbol{v}$ 移动导体。自由电荷上将存在与 $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$成比例的洛伦兹力，这进一步更新电流密度中的附加项以使破坏互易关系变的可能：

$$J \to J + \sigma(E + v \times B).$$

A2. BREAKING RECIPROCITY RELATIONSHIP 打破互易关系

A2	Express the possibly non-zero $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$ as a volume integral in terms of the vector potentials $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ and the conductor velocity $\boldsymbol{v}$. 将可能非零的 $\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2)$表示为根据矢量势$\boldsymbol{A}_1$ 和$\boldsymbol{A}_2$ 以及导体速度$\boldsymbol{v}$ 的体积积分。	3 points 3 分

Solution:

The Ampère's law becomes

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Then, the additional part to $\nabla \cdot (\boldsymbol{H}_1 \times \boldsymbol{A}_2 - \boldsymbol{H}_2 \times \boldsymbol{A}_1)$ is

$$\sigma(\boldsymbol{E}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{E}_2 \cdot \boldsymbol{A}_1) + \sigma(\boldsymbol{v} \times \boldsymbol{B}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{v} \times \boldsymbol{B}_2 \cdot \boldsymbol{A}_1)$$

The volume integration of the first term goes to zero for $\nabla \times \boldsymbol{E}_i = \boldsymbol{0}$ and decay of the boundary terms as r approaching infinity.

The second term is

$$\begin{aligned} &\sigma\boldsymbol{v} \cdot (\boldsymbol{B}_1 \times \boldsymbol{A}_2 - \boldsymbol{B}_2 \times \boldsymbol{A}_1) \\ &= \sigma\boldsymbol{v} \cdot \big((\nabla \times \boldsymbol{A}_1) \times \boldsymbol{A}_2 - (\nabla \times \boldsymbol{A}_2) \times \boldsymbol{A}_1\big) \end{aligned}$$

Therefore

$$\boldsymbol{m}_1 \cdot \boldsymbol{B}_2(\boldsymbol{r}_1) - \boldsymbol{m}_2 \cdot \boldsymbol{B}_1(\boldsymbol{r}_2) = \int \sigma\boldsymbol{v} \cdot \big(\boldsymbol{A}_2 \times (\nabla \times \boldsymbol{A}_1) - \boldsymbol{A}_1 \times (\nabla \times \boldsymbol{A}_2)\big) dV$$

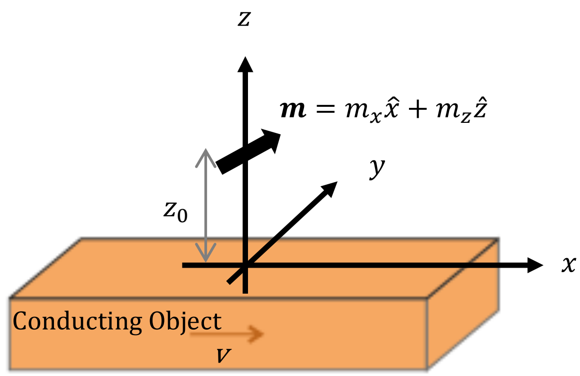
You can complete part B and C without part A.

你可以在没有 A 部分的情况下完成 B 部分和 C 部分。

B. MAGNETIC DIPOLES ON A MOVING PERFECT CONDUCTOR 运动理想导体上的磁偶极子

In this part, we first obtain the magnetic field of a single magnetic dipole on a moving conductor, which is a perfect conductor (i.e. the electrical conductivity $\sigma = \infty$) here, occupying $z \leq 0$ and is moving with a velocity v in the positive x-direction. This is defined as the laboratory frame, as shown in the figure below. The magnetic dipole, situated at $(x, y, z) = (0,0, z_0)$ on top of the conductor, has a magnetic moment $\boldsymbol{m} = m_x\hat{x} + m_z\hat{z}$ with zero component in the y-direction.

在这一部分中，我们首先得到运动导体上单个磁偶极子的磁场，这里是理想导体 (即电导率 $\sigma = \infty$)，占据 z≤0 并且在正 *x* 方向上以速度 v 运动。 这被定义为实验室坐标系，如下图所示。 位于导体顶部 $(x, y, z) = (0,0, z_0)$ 处的磁偶极子写为 $\boldsymbol{m} = m_x\hat{x} + m_z\hat{z}$，在 *y* 方向上没有分量。



In the moving frame at a velocity $\boldsymbol{v} = v\hat{x}$ with respect to the laboratory frame, the object is simply a perfect conductor at rest, giving us a convenience to find the magnetic fields generated by the magnetic dipole. The coordinates in the moving frame, denoted as (x', y', z', t'), is transformed from the coordinates in the laboratory frame (x, y, z, t) through the Lorentz transformation:

在相对于实验室坐标系以速度 $\boldsymbol{v} = v\hat{x}$ 运动的坐标系中，物体只是一个静止的理想导体，使我们更方便地找到磁偶极子产生的磁场。 移动坐标系中的坐标，表示为 (x', y', z', t')，由实验室坐标系(x, y, z, t) 中的坐标通过洛伦兹变换转换而来：

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad ct' = \gamma(ct - vx/c)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and c is the speed of light.

其中 $\gamma = 1/\sqrt{1 - v^2/c^2}$， c 是光速。

The magnetic and electric fields ($\boldsymbol{B}'$ and $\boldsymbol{E}'$) in the moving frame are related to the fields ($\boldsymbol{B}$ and $\boldsymbol{E}$) in the laboratory frame by the Lorentz transformation,

移动坐标系中的磁场和电场（$\boldsymbol{B}'$和 $\boldsymbol{E}'$）与实验室坐标系中的场（$\boldsymbol{B}$ 和 $\boldsymbol{E}$）相关，可由洛伦兹变换给出:

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix} + \frac{v\gamma}{c^2}\begin{pmatrix} 0 \\ E_z \\ -E_y \end{pmatrix}, \qquad \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma E_y \\ \gamma E_z \end{pmatrix} - v\gamma \begin{pmatrix} 0 \\ B_z \\ -B_y \end{pmatrix}$$

We further simplify the problem by removing the conductor at the moment.

我们现在通过移除导体来进一步简化问题。

B1. MAGNETIC FIELD OF A MOVING MAGNETIC DIPOLE IN FREESPACE 自由空间中运动磁偶极子的磁场

B1	What is the magnetic field $\boldsymbol{B}'(x', y', z', t')$ from the dipole for an observer moving at velocity v with respect to the laboratory frame? In this part, we only consider a magnetic dipole $\boldsymbol{m} = m_x\hat{x}$ pointing in the x-direction and there is no conductor below the dipole yet. Please express your answer in the coordinates of the moving frame. 请找出相对于实验室坐标系以速度v 移动的观察者的偶极子磁场 $\boldsymbol{B}'(x', y', z', t')$。 在这一部分中，我们只考虑指向 x 方向的磁偶极子$\boldsymbol{m} = m_x\hat{x}$ 并且偶极子下方还没有导体。 请用移动坐标系的坐标表达你的答案。	3 point 3 分

Solution:

The B-field for a magnetic dipole $\boldsymbol{m} = m_x\hat{x}$ held at $z_0\hat{z}$ is

$$\frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{r} - z_0\hat{z})(\boldsymbol{r} - z_0\hat{z}) \cdot \boldsymbol{m}}{|\boldsymbol{r} - z_0\hat{z}|^5} - \frac{\boldsymbol{m}}{|\boldsymbol{r} - z_0\hat{z}|^3}\right)$$

For m_x dipole, we have

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_x}{4\pi(x^2+y^2+(z-z_0)^2)^{5/2}} \begin{pmatrix} 2x^2 - y^2 - (z-z_0)^2 \\ 3xy \\ 3x(z-z_0) \end{pmatrix}$$

In the moving frame, the fields transform to

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix}$$

$$= \frac{\mu_0 m_x}{4\pi(\gamma^2(x'+vt')^2+y'^2+(z'-z_0)^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2 - y'^2 - (z'-z_0)^2 \\ 3\gamma^2(x'+vt')y' \\ 3\gamma^2(x'+vt')(z'-z_0) \end{pmatrix}$$

where

$$x = \gamma(x'+vt'), y = y', z = z', ct = \gamma(ct' + vx'/c)$$

with $\gamma = 1/\sqrt{1-v^2/c^2}$. We also have $x' = -vt' + x/\gamma$. Full marks if the answer is given in non-relativistic limit $\gamma \to 1$.

Next, we introduce the perfect conductor below the dipole. The perfect metal has a planar interface at $z = 0$ while the dipole is placed at a distance z_0 above the metal. The metal is being moved at a velocity v in the positive x-direction.

接下来，我们引入偶极子下方的理想导体。 理想导体在 z=0 处有一个平面界面，而偶极子位于金属上方 z_0 处。 金属在正 x 方向上以速度 v 移动。

B2. MAGNETIC FIELD OF A MAGNETIC DIPOLE ON A MOVING CONDUCTOR (I) 运动导体上磁偶极子的磁场（1）

B2	Please express the magnetic field of the magnetic dipole $\boldsymbol{m} = m_x\hat{x}$ on top of a moving perfect conductor (i.e $\boldsymbol{B}(x, y, 0^+)$) in terms of the laboratory frame coordinate. Please also verify the boundary condition of the magnetic field in the moving frame at which the conductor at rest. Hint: adopt the method of images and assume both electric and magnetic fields inside a perfect conductor is zero at a nearly-zero frequency. 请用实验室坐标系表示在移动中的理想导体上磁偶极子$\boldsymbol{m} = m_x\hat{x}$ 的磁场(即 $\boldsymbol{B}(x, y, 0^+)$)。 还请验证导体静止时移动坐标系中的磁场边界条件。提示：采用镜像法。假定在接近零频率时完美导体里电场和磁场为零。	4 points 4 分

Solution:

We adopt the method of images by having a magnetic dipole m_x at mirror location at $-z_0\hat{z}$. In the moving frame with velocity v with respect to the laboratory frame, the conductor is at rest, and we can apply the normal boundary condition $B'_z = 0$ on the surface of the perfect conductor.

In the laboratory frame (dipole at rest), the total field is

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_x}{4\pi} \frac{1}{(x^2+y^2+(z-z_0)^2)^{5/2}} \begin{pmatrix} 2x^2 - y^2 - (z-z_0)^2 \\ 3xy \\ 3x(z-z_0) \end{pmatrix} + \frac{\mu_0 m_x}{4\pi} \frac{1}{(x^2+y^2+(z+z_0)^2)^{5/2}} \begin{pmatrix} 2x^2 - y^2 - (z+z_0)^2 \\ 3xy \\ 3x(z+z_0) \end{pmatrix}$$

for $z \geq 0$.

The total field in the moving frame is

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix} = \frac{\mu_0 m_x}{4\pi} \frac{1}{(\gamma^2(x'+vt')^2 + y'^2 + (z'-z_0)^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2 - y'^2 - (z'-z_0)^2 \\ 3\gamma^2(x'+vt')y' \\ 3\gamma^2(x'+vt')(z'-z_0) \end{pmatrix}$$

$$+ \frac{\mu_0 m_x}{4\pi} \frac{1}{(\gamma^2(x'+vt')^2 + y'^2 + (z'+z_0)^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2 - y'^2 - (z'+z_0)^2 \\ 3\gamma^2(x'+vt')y' \\ 3\gamma^2(x'+vt')(z'+z_0) \end{pmatrix}$$

for $z' \geq 0$.

At $z' = 0^+$, we have

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \frac{\mu_0 m_x}{2\pi} \frac{1}{(\gamma^2(x'+vt')^2 + y'^2 + z_0^2)^{5/2}} \begin{pmatrix} 2\gamma^2(x'+vt')^2 - y'^2 - z_0^2 \\ 3\gamma^2(x'+vt')y' \\ 0 \end{pmatrix}$$

satisfying boundary condition on surface of conductor $B'_z = 0$ in justifying the solution from the method of image.

The magnetic field in the laboratory frame (for the case of no electric fields in the laboratory frame) is

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} B'_x \\ B'_y/\gamma \\ B'_z/\gamma \end{pmatrix} = \frac{\mu_0 m_x}{2\pi} \frac{1}{(x^2 + y^2 + z_0^2)^{5/2}} \begin{pmatrix} (2x^2 - y^2 - z_0^2)/\gamma \\ 3xy \\ 0 \end{pmatrix}$$

Full marks are given if answers are expressed in either laboratory frame or moving frame coordinates.

For a perfect metal, the mirror dipole of (m_x, m_y, m_z) is $(m_x, m_y, -m_z)$ by definition. However, it is required to verify the boundary condition in the moving frame as stated.

B3. MAGNETIC FIELD OF A MAGNETIC DIPOLE ON A MOVING CONDUCTOR (2) 运动导体上磁偶极子的磁场（2）

B3	What is the magnetic field from the magnetic dipole on top of a moving perfect conductor (i.e $\boldsymbol{B}(x, y, 0^+)$) in the laboratory frame if the magnetic dipole above the moving conductor is changed to $\boldsymbol{m} = m_z\hat{z}$, pointing in the positive z direction? 如果运动导体上方的磁偶极子转为$\boldsymbol{m} = m_z\hat{z}$，指向正 z 方向，那么请找出实验室坐标系中磁偶极子在移动中的理想导体上的磁场(即 $\boldsymbol{B}(x, y, 0^+)$)？	3 points 3 分

Solution:

For a m_z dipole, we have the magnetic field in the laboratory frame as

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_z}{4\pi(x^2 + y^2 + (z-z_0)^2)^{5/2}} \begin{pmatrix} 3x(z-z_0) \\ 3y(z-z_0) \\ -x^2 - y^2 + 2(z-z_0)^2 \end{pmatrix}$$

Putting a mirror dipole $-m_z$ results the total field as

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_z}{4\pi(x^2 + y^2 + (z-z_0)^2)^{5/2}} \begin{pmatrix} 3x(z-z_0) \\ 3y(z-z_0) \\ -x^2 - y^2 + 2(z-z_0)^2 \end{pmatrix} - \frac{\mu_0 m_z}{4\pi(x^2 + y^2 + (z+z_0)^2)^{5/2}} \begin{pmatrix} 3x(z+z_0) \\ 3y(z+z_0) \\ -x^2 - y^2 + 2(z+z_0)^2 \end{pmatrix}$$

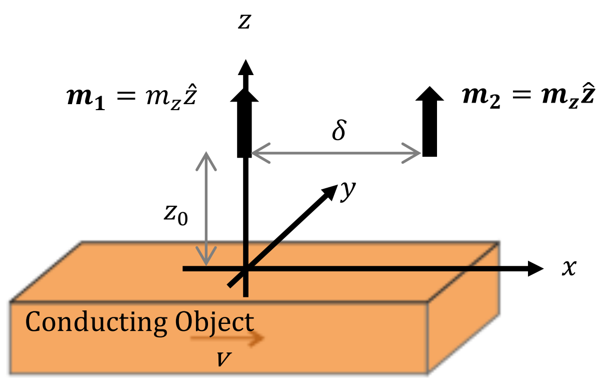
At $z = 0$, we have

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = -\frac{3z_0\mu_0 m_z}{2\pi(x^2+y^2+z_0^2)^{5/2}}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

In principle, the fields are transformed to the moving frame at which the boundary condition is verified, as we have done in part B.2, but we have skipped here for brevity.

Now, we consider the two situations: one with magnetic dipole moment $\boldsymbol{m}_1$ at $\boldsymbol{r}_1 = (0,0,z_0)$ and another magnetic dipole moment $\boldsymbol{m}_2$ at $\boldsymbol{r}_2 = (\delta, 0, z_0)$. When one of them is turned on, another one is turned off.

现在，我们考虑两种情况：一种是 $\boldsymbol{r}_1 = (0,0,z_0)$ 处的磁偶极矩 $\boldsymbol{m}_1$，另一种是 $\boldsymbol{r}_2 = (\delta, 0, z_0)$ 处的磁偶极矩 $\boldsymbol{m}_2$。当其中一个打开时，另一个被关闭。



Now, we define the two problems to solve next. For the *first* problem, suppose the two magnetic dipoles moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ point in the z-direction with same size m_z, as shown in the above figure. Dipole moment $\boldsymbol{m}_1$ imposes a magnetic field of z-component $B_z(\delta)$ on $\boldsymbol{r}_2 = (x_2, 0, z_0)$ and dipole $\boldsymbol{m}_2$ imposes magnetic field $B_z(-\delta)$ on $\boldsymbol{r}_1 = (x_1, 0, z_0)$ according to the same function $B_z(x - x_i)$. We then define a reciprocity figure-of-merit

$$\mathcal{R} = \frac{B_z(\delta) - B_z(-\delta)}{B_z(\delta) + B_z(-\delta)}.$$

When $\mathcal{R} = 0$, reciprocity is satisfied. Reciprocity is broken when $\mathcal{R}$ deviates from value zero.

现在，我们定义接下来要解决的两个问题。对于第一个问题，假设两个磁偶极子$\boldsymbol{m}_1$和$\boldsymbol{m}_2$都指向 z 方向，大小m_z相同，如上图所示。偶极子 $\boldsymbol{m}_1$ 根据函数 $B_z(x - x_i)$ 将 z 分量 $B_z(\delta)$ 的磁场施加到 $\boldsymbol{r}_2 = (x_2, 0, z_0)$上，偶极子 $\boldsymbol{m}_2$ 将磁场 $B_z(-\delta)$ 施加到 $\boldsymbol{r}_1 = (x_1, 0, z_0)$ 上。然后我们定义互易功值

$$\mathcal{R} = \frac{B_z(\delta) - B_z(-\delta)}{B_z(\delta) + B_z(-\delta)}.$$

当$\mathcal{R} = 0$时，满足互易性。当 $\mathcal{R}$ 偏离零值时，互易性被打破。

For the *second* problem, suppose we change the pointing direction for $\boldsymbol{m}_1$ to the positive x-direction with magnitude remaining the same. The magnitude and direction of $\boldsymbol{m}_2$ are not changed. In this case, the reciprocity merit is defined as $\mathcal{R} = \big(B_{1z}(\delta) - B_{2x}(-\delta)\big)/\big(B_{1z}(\delta) + B_{2x}(-\delta)\big)$.

对于第二个问题，假设我们将 $\boldsymbol{m}_1$的指向更改为正 x 方向而大小保持不变。$\boldsymbol{m}_2$方向和大小不变。在这种情况下，互易功值定义为 $\mathcal{R} = \big(B_{1z}(\delta) - B_{2x}(-\delta)\big)/\big(B_{1z}(\delta) + B_{2x}(-\delta)\big)$。

B4. RECIPROCITY MERIT FOR DIPOLES ON A PERFECT CONDUCTOR 理想导体上偶极子的互易功值

B4	Find the reciprocity merit $\mathcal{R}$ for the two defined problems about magnetic dipoles on the moving perfect conductor. 求出关于以上两个问题运动理想导体上磁偶极子的互易功值$\mathcal{R}$。	3 points 3 分

Solution:

For the first case, the magnetic field from dipole 1 is

$$B_z(x,0,z_0) = -\frac{\mu_0 m_z}{4\pi|x|^5}x^2 - \frac{\mu_0 m_z}{4\pi(x^2+4z_0^2)^{5/2}}(-x^2+8z_0^2)$$

which is even in x. Then, we have $\mathcal{R}=0$.

For the second case, the magnetic field from dipole 1 on dipole 2 is

$$B_{1z}(\delta) = \frac{\mu_0 m_x}{4\pi}\frac{6xz_0}{(x^2+4z_0^2)^{5/2}}$$

The magnetic field from dipole 2 on dipole 1 is

$$B_{2x}(-\delta) = -\frac{\mu_0 m_z}{4\pi}\frac{6xz_0}{(x^2+4z_0^2)^{5/2}}$$

with $m_z = m_x$.

They have the same values and hence $\mathcal{R}=0$.

In fact, for a moving perfect metal, although breaking the time-reversal symmetry, is still not able to generate the non-reciprocal coupling. We need dissipation with a finite conductivity.

C. MAGNETIC DIPOLES ON A MOVING CONDUCTOR WITH FINITE CONDUCTIVITY 具有有限电导率的移动导体上的磁偶极子

In this part, we move to a more realistic situation that the conductor is a metal. It has a large but finite conductivity σ (in unit of $\Omega^{-1}m^{-1}$), deviating from the perfect conductor condition. Current density in the conductor is given by $\boldsymbol{J}=\sigma\boldsymbol{E}$. We also assume that the current on the conductor surface is confined by a skin depth d of small thickness so that the electric and magnetic fields cannot penetrate beyond the skin depth from the conductor surface. Then, the surface current density can be written as $(\sigma d)\boldsymbol{E}$. We further take the approximation that d is just a constant. We only consider the two dipoles pointing in the positive z-direction with same size m_z in this part.

在这一部分中，我们转向一个更现实的情况，即金属导体具有较大但有限的电导率 σ（单位为 $\Omega^{-1}m^{-1}$），偏离理想导体条件。 导体中的电流密度由 $\boldsymbol{J}=\sigma\boldsymbol{E}$ 给出。 我们还假设电流只在导体表面厚度较小的趋肤深度 d 内流动，电场和磁场不能从导体表面穿透超过趋肤深度。 因此表面电流密度可以写成 $(\sigma d)\boldsymbol{E}$。我们进一步假定 d 为一个常数。 在这部分我们只考虑两个偶极子都指向正 z 方向，大小 m_z相同。

Again, we need to solve the magnetic field from only one dipole at $(0,0,z_0)$ first. In fact, the surface current profile generated on the surface of conductor cannot be easily solved without adopting a numerical solver. Instead, we can approach the problem by extending the method of image as an approximation. In this case, we would like to have a point-like multipolar source at the image position $(0,0,-z_0)$ in order to give as closely as possible the same reflected field generated from the surface current. For the current case of $\boldsymbol{m}=m_z\hat{z}$, we put an image magnetic dipole with given form of magnetic moment $m_x^{(r)}\hat{x}+m_z^{(r)}\hat{z}$ and electric moment $p_y^{(r)}\hat{y}$ at the same location $(0,0,-z_0)$. The mirrored magnetic dipole is now relaxed to have both magnetic and electric components while neglecting the higher order multipoles. The size of these dipole moments are yet to be determined.

同样，我们需要先求解 $(0,0,z_0)$ 处只有一个偶极子的磁场。 实际上，导体表面产生的表面电流分布不采用数值求解器是无法精确求解的。 这里，我们尝试通过电像法来近似解决这个问题。 在这种情况下，我们希望在镜像位置 $(0,0,-z_0)$ 处有一个点状多极源，以便尽可能地提供相同的从表面电流生成的反射场。 对于当前 $\boldsymbol{m} = m_z\hat{z}$ 的情况，我们将镜像磁偶极子设置在$(0,0,-z_0)$ 处，它拥有给定的表达式：磁矩为 $m_x^{(r)}\hat{x} + m_z^{(r)}\hat{z}$ 、电矩为 $p_y^{(r)}\hat{y}$。 镜像磁偶极子现在同时具有磁和电分量，同时忽略高阶多极子。 这些偶极矩的大小有待确定。

C1. GENERALIZED METHOD OF IMAGES 广义电像法

C1	Find the magnetic field $\boldsymbol{B}'(x',y',0^+)$ and electric field $\boldsymbol{E}'(x',y',0^+)$ on the conductor surface in the moving frame. Express your answer in m_z, $m_x^{(r)}$, $m_z^{(r)}$ and $p_y^{(r)}$. You can use either the moving frame or laboratory frame coordinates. Do not need to solve the mirrored dipole moments yet. 求移动坐标系导体表面的磁场$\boldsymbol{B}'(x',y',0^+)$ 和电场$\boldsymbol{E}'(x',y',0^+)$。 请用$m_z$、$m_x^{(r)}$、$m_z^{(r)}$ 和$p_y^{(r)}$ 表达你的答案。可以使用移动坐标或实验室坐标表达你的答案。 暂时不需要求解镜像偶极矩。	5 points 5 分

Solution:

The magnetic field comes from the magnetic dipole m_z at $(0,0,z_0)$, magnetic dipole $m_x^{(r)}\hat{x} + m_z^{(r)}\hat{z}$ at $(0,0,-z_0)$ and a small contribution from electric dipole $p_y^{(r)}$ at $(0,0,-z_0)$ due to the change of reference frame.

The dipole fields on metal surface in the laboratory frame are

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0 m_z}{4\pi(x^2+y^2+(z-z_0)^2)^{5/2}} \begin{pmatrix} 3x(z-z_0) \\ 3y(z-z_0) \\ -x^2-y^2+2(z-z_0)^2 \end{pmatrix} + \frac{\mu_0 m_z^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}} \begin{pmatrix} 3x(z+z_0) \\ 3y(z+z_0) \\ -x^2-y^2+2(z+z_0)^2 \end{pmatrix}$$
$$+ \frac{\mu_0 m_x^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}} \begin{pmatrix} 2x^2-y^2-(z+z_0)^2 \\ 3xy \\ 3x(z+z_0) \end{pmatrix}$$

at $z=0$, giving

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{\mu_0}{4\pi(x^2+y^2+z_0^2)^{5/2}} \begin{pmatrix} 3xz_0\left(m_z^{(r)}-m_z\right) + (2x^2-y^2-z_0^2)m_x^{(r)} \\ 3yz_0\left(m_z^{(r)}-m_z\right) + 3xym_x^{(r)} \\ (2z_0^2-x^2-y^2)\left(m_z^{(r)}+m_z\right) + 3xz_0 m_x^{(r)} \end{pmatrix}$$

We also have

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{\mu_0 c^2 p_y^{(r)}}{4\pi(x^2+y^2+z_0^2)^{5/2}} \begin{pmatrix} 3xy \\ -x^2+2y^2-z_0^2 \\ 3yz_0 \end{pmatrix}$$

In the moving frame, the fields are transformed by

$$\begin{pmatrix} B_x' \\ B_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma B_y \\ \gamma B_z \end{pmatrix} + \frac{v\gamma}{c^2} \begin{pmatrix} 0 \\ E_z \\ -E_y \end{pmatrix}$$
$$\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma E_y \\ \gamma E_z \end{pmatrix} - v\gamma \begin{pmatrix} 0 \\ B_z \\ -B_y \end{pmatrix}$$

Then,

$$\begin{pmatrix} B_x' \\ B_y' \\ B_z' \end{pmatrix} / \frac{\mu_0}{4\pi(x^2+y^2+z_0^2)^{5/2}} = \begin{pmatrix} 3xz_0\left(m_z^{(r)} - m_z\right) + (2x^2 - y^2 - z_0^2)m_x^{(r)} \\ 3yz_0\gamma\left(m_z^{(r)} - m_z + v\gamma p_y^{(r)}\right) + 3xym_x^{(r)} \\ (2z_0^2 - x^2 - y^2)\gamma\left(m_z^{(r)} + m_z\right) + 3xz_0 m_x^{(r)} - (-x^2 + 2y^2 - z_0^2)v\gamma^2 p_y^{(r)} \end{pmatrix}$$

$$\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix} / \frac{\mu_0}{4\pi(x^2+y^2+z_0^2)^{5/2}} = \begin{pmatrix} 3xyc^2 p_y^{(r)} \\ (-x^2 + 2y^2 - z_0^2)\gamma c^2 p_y^{(r)} - v\gamma(2z_0^2 - x^2 - y^2)\left(m_z^{(r)} + m_z\right) - 3xz_0 v\gamma m_x^{(r)} \\ 3y\gamma\left(z_0 c^2 p_y^{(r)} + z_0 v\left(m_z^{(r)} - m_z\right) + xvm_x^{(r)}\right) \end{pmatrix}$$

Full marks will be given for fields expressed either in lab frame or moving frame coordinates.

C2. FINDING THE MIRRORED DIPOLE MOMENTS 计算镜像偶极矩

C2	Find $m_x^{(r)}$, $m_z^{(r)}$ and $p_y^{(r)}$ in response to a given m_z. The mirror dipole gives the same reflected field generated by the surface current on the conductor. As approximation, only apply the boundary condition (in the moving frame) on the surface current along the y-direction, which is the dominant current than the one along the x-direction. It may be useful to express the answers in term of the dimensionless parameter $\kappa = \mu_0 v\gamma\sigma d \gg 1$. 用给定的$m_z$表示 $m_x^{(r)}$、$m_z^{(r)}$ 和 $p_y^{(r)}$。 镜像偶极子给出了与导体表面电流产生的相同的反射场。 作为近似，请仅将边界条件（在移动坐标系中）应用于沿 y 方向的表面电流，该电流比沿 x 方向的电流占主导地位。可以考虑用无量纲参数$\kappa = \mu_0 v\gamma\sigma d \gg 1$ 表达答案。	5 points 5 分

Solution:

The surface current is given by

$$\boldsymbol{J}_s' = d\,\sigma\,\boldsymbol{E}' = \hat{z} \times \boldsymbol{B}'/\mu_0$$

at $z' = 0$. The second equality comes from Ampère's law with the assumption that the field decays to negligible values beyond skin depth. The dominant current is along the y-direction. We have $B_x' = \mu_0 d\,\sigma\,E_y'$:

$$3xz_0\left(m_z^{(r)} - m_z\right) + (2x^2 - y^2 - z_0^2)m_x^{(r)}$$
$$= \kappa\left((-x^2 + 2y^2 - z_0^2)c^2 p_y^{(r)}/v - (2z_0^2 - x^2 - y^2)\left(m_z^{(r)} + m_z\right) - 3xz_0 m_x^{(r)}\right)$$

where $\kappa = \mu_0 v\gamma\sigma d$. Equating coefficients for the different powers of x and y give

$$2m_x^{(r)} = \kappa\left(m_z^{(r)} + m_z - c^2 p_y^{(r)}/v\right)$$

$$-m_x^{(r)} = \kappa\left(m_z^{(r)} + m_z + 2c^2 p_y^{(r)}/v\right)$$
$$m_z^{(r)} - m_z = -\kappa m_x^{(r)}$$

and hence

$$\frac{m_x^{(r)}}{m_z} = \frac{2\kappa}{1+\kappa^2} \cong \frac{2}{\kappa}$$
$$\frac{m_z^{(r)}}{m_z} = \frac{1-\kappa^2}{1+\kappa^2} \cong -1 + \frac{2}{\kappa^2}$$
$$\frac{p_y^{(r)}}{m_z} = -\frac{v}{c^2}\frac{2}{1+\kappa^2} \cong -\frac{v}{c^2}\frac{2}{\kappa^2}$$

where

$$\kappa = \mu_0 v\gamma\sigma d.$$

We have assumed a large κ limit so that the material deviates a bit from the perfect metal.

C3. RECIPROCITY MERIT FOR DIPOLES ON A CONDUCTOR OF FINITE CONDUCTIVITY
有限电导率导体上偶极子的互易功值

C3	Find the reciprocity merit $\mathcal{R}$ for the two identical dipoles m_z displaced by δ (with $\boldsymbol{r}_1 = (0,0,z_0)$, $\boldsymbol{r}_2 = (\delta, 0, z_0)$)in the x-direction. 找出在 x 方向上位移δ （$\boldsymbol{r}_1 = (0,0,z_0), \boldsymbol{r}_2 = (\delta, 0, z_0)$）的两个相同偶极子 m_z 的互易功价值$\mathcal{R}$。	3 point 3 分

Solution:

The magnetic field has contributions from the original m_z dipole and the mirror magnetic dipole:

$$B_z(x,y,z) = \frac{\mu_0 m_z}{4\pi(x^2+y^2+(z-z_0)^2)^{5/2}}(-x^2-y^2+2(z-z_0)^2) + \frac{\mu_0 m_z^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}}(-x^2-y^2+2(z+z_0)^2) + \frac{\mu_0 m_x^{(r)}}{4\pi(x^2+y^2+(z+z_0)^2)^{5/2}}\left(3x(z+z_0)\right)$$

$$B_z(\delta,0,z_0) = \frac{\mu_0 m_z}{4\pi(\delta^2+4z_0^2)^{5/2}}\left(-\frac{(\delta^2+4z_0^2)^{5/2}}{|\delta|^3} + \frac{m_z^{(r)}}{m_z}(8z_0^2-\delta^2) + \frac{m_x^{(r)}}{m_z}6z_0\delta\right)$$

Then

$$\mathcal{R} = \frac{B_z(\delta,0,z_0) - B_z(-\delta,0,z_0)}{B_z(\delta,0,z_0) + B_z(-\delta,0,z_0)} = sign(\delta)\frac{\frac{m_x^{(r)}}{m_z}6|\delta/z_0|^4}{-(4+|\delta/z_0|^2)^{5/2} + \frac{m_z^{(r)}}{m_z}(8-|\delta/z_0|^2)|\delta/z_0|^3}$$

When the material is approaching a perfect metal, we have $m_z^{(r)} \cong -m_z$ to be substituted into the denominator, giving

$$\mathcal{R} \cong -sign(\delta)\frac{m_x^{(r)}}{m_z}\frac{6|\delta/z_0|^4}{(4+|\delta/z_0|^2)^{5/2} + (8-|\delta/z_0|^2)|\delta/z_0|^3}$$

By further substituting last part's answer, we obtain

$$\mathcal{R} \cong -\frac{sign(\delta)}{\kappa}\frac{12|\delta/z_0|^4}{(4+|\delta/z_0|^2)^{5/2} + (8-|\delta/z_0|^2)|\delta/z_0|^3}$$

Full marks can be given without the final substitution.

Reference: J. Prat-Camps, P. Maurer, G. Kichmair, and O. Romero-Isart, Phys. Rev. Lett. 121, 213903 (2018).