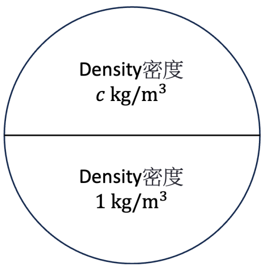
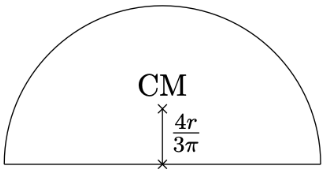
Pan Pearl River Delta Physics Olympiad 2024
2024 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 18th Feb 2024)

Please fill in your final answers to all problems on the answer sheet.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. [10 points] We consider an inhomogeneous cylinder whose have are mode of two materials of different densities. The cylinder has radius r and length L and its cross section is shown in Fig. 1a. The bottom half made of a material whose density is 1 kg/m3 while the upper half is mode of a material whose density is c kg/m3, where c is a parameter $0 < c < 1$. In the problem, we can use the fact that for a half-cylinder of radius r, the center of mass (CM) is located at a distance of $\frac{4r}{3\pi}$ from the axis of the half-cylinder, as shown in Fig. 1b.
1. [10 分] 我们考虑一个不均匀的圆柱体，其由两种密度不同的材料构成。圆柱体的半径为 r，长度为L，其横截面如图 1a 所示。下半部分由密度为$1\ \mathrm{kg/m^3}$ 的材料制成，而上半部分由密度为$c\ \mathrm{kg/m^3}$ 的材料制成，其中c是一个参数，$0 < c < 1$。在问题中，我们可以利用这样一个事实，对于一个半径为 r 的半圆柱体，其质心 (CM)位于半圆柱体轴线距离为 $\frac{4r}{3\pi}$ 的位置，如图 1b 所示。





(ai) [2] Compute the total mass M of the entire inhomogeneous cylinder, and the distance d between the geometrical center and the center of mass of the entire cylinder. Express the answer is terms of r, L, c.
(aii) [2] Compute the moment of inertia I of the entire inhomogeneous cylinder with respect to its geometrical axis. Express the answers in terms of r, L, c.

(b) [2] The geometrical axis of the cylinder is fixed in a horizontal position, but the cylinder is free to rotate without any friction around the axis. If the cylinder oscillates around its stable equilibrium position with small amplitude, calculate the period of oscillation of the cylinder. Express the answer in terms of M, I, r, d and the gravitational acceleration g.

Now we assume that the cylinder is completely free to move on a horizontal table under the gravity. We assume that the coefficient of static friction between the cylinder and the table is infinite, such that the cylinder cannot slide. Suppose that at time $t = 0$, the cylinder is in its equilibrium position with an initial angular velocity ω_0.
(ci) [2] If ω_0 is sufficiently small, the cylinder will undergo a period motion around its stable equilibrium. What is the period of oscillation if the amplitude of the oscillation is small? Express the answer in terms of M, I, r, d, g.
(cii) [2] What is the minimum value of ω_0 that allows the cylinder to roll forever in the same direction. Express the answer in terms of M, I, r, d.

(ai) [2] 计算整个不均匀圆柱体的总质量 M 和几何中心与质心之间的距离 d。用r，L，c 表示答案。
(aii) [2] 计算整个不均匀圆柱体相对于其几何轴的转动惯量 I。用 r，L，c 表示答案。

(b) [2] 圆柱体的几何轴固定在水平位置，但圆柱体围绕几何轴作自由旋转，没有任何摩擦。如果圆柱体在其稳定平衡位置周围振荡，并且振幅很小，计算圆柱体的振荡周期。用M，I，r，d 和重力加速度 g 表示答案。

现在我们假设圆柱体在重力下完全自由地在水平枱面上移动。我们假设圆柱体和枱面之间的静摩擦系数是无限大的，因此圆柱体无法滑动。假设在时刻 $t = 0$ 时，圆柱体处于其平衡位置，并具有初始角速度 ω_0。
(ci) [2] 如果 ω_0 足够小，圆柱体将围绕其稳定平衡进行周期运动。如果振幅很小，振荡周期是多少？用 M，I，r，d，g表示答案。

(cii) [2] 让圆柱体永远向同一方向滚动的最小 ω_0值是多少？用M ，I ，r ，d 表示答案。

Solution:

(ai)

$$M = \frac{\pi r^2 L}{2}(1+c)$$

$$d = \frac{\frac{\pi r^2 L}{2}\left(\frac{4r}{3\pi} - \frac{4r}{3\pi}c\right)}{\frac{\pi r^2 L}{2}(1+c)} = \frac{4r}{3\pi}\left(\frac{1-c}{1+c}\right)$$

(aii) The moment of inertia of a cylinder of mass M and radius r is $\frac{1}{2}Mr^2$

$$I = \frac{1}{2}\left(\frac{\pi r^2 L}{2}\right)(1+c)r^2 = \frac{\pi r^4 L}{4}(1+c)$$

(b) Newton's 2nd law gives,

$$\tau = I\ddot{\theta}$$

The torque w.r.t. the geometrical axis is

$$\tau = -Mgd\sin\theta$$
$$I\ddot{\theta} = -Mgd\sin\theta$$
$$\ddot{\theta} = -\frac{Mgd}{I}\sin\theta$$
$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgd}}$$

(ci) We consider the equation of motion w.r.t. the contact point on the ground,

$$\tau = I_{con}\ddot{\theta}$$

Where the torque is the same as in part b,

$$\tau = -Mgd\sin\theta \approx -Mgd\theta$$

The moment of inertia w.r.t. the contact is given by

$$I_{con} = I_{cm} + M(r-d)^2$$
$$I = I_{cm} + Md^2$$
$$\Rightarrow I_{con} = I + M((r-d)^2 - d^2) = I + M(r^2 - 2rd)$$

$$\Rightarrow \ddot{\theta} = -\frac{Mgd}{I_{con}}\theta$$

The period

$$T = 2\pi\sqrt{\frac{I_{con}}{Mgd}} = 2\pi\sqrt{\frac{I + M(r^2 - 2rd)}{Mgd}}$$

You can see that as $d \to 0$, the period $T \to \infty$ which is expected.

(Method 2):

The Lagrangian is

$$L = T - U = \frac{1}{2}I_{con}\dot{\theta}^2 - \frac{1}{2}Mgd\theta^2$$

The EOM is given by,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow I_{con}\ddot{\theta} + Mgd\theta = 0$$
$$\Rightarrow \ddot{\theta} = -\frac{Mgd}{I_{con}}\theta$$
$$\Rightarrow T = 2\pi\sqrt{\frac{I_{con}}{Mgd}}2\pi\sqrt{\frac{I + M(r^2 - 2rd)}{Mgd}}$$

(cii) Since the cylinder can only roll without sliding, the total energy is conserved. In order to escape from the oscillation, the cylinder must have enough kinetic energy to overcome the gravitational PE when its CM is directly above the geometrical axis.

$$\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega_0^2 + Mg(r-d) = Mg(r+d)$$

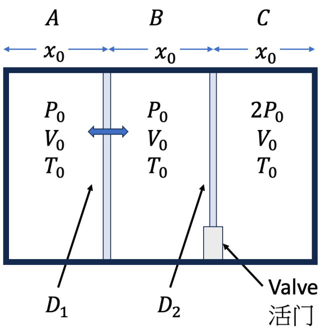
$$\Rightarrow \frac{1}{2}M\omega_0^2(r-d)^2 + \frac{1}{2}(I - Md^2)\omega_0^2 = 2Mgd$$

$$\Rightarrow \omega_0 = \sqrt{\frac{4Mgd}{M(r-d)^2 + (I - Md^2)}}$$

We can see that as $d \to 0, \omega_0 \to 0$. It implies that there is no stable equilibrium in this limit and the cylinder will move forever given any infinitesimal initial push.

2. [10 points] A closed container is divided into three compartments, A, B, and C, by two partitions, D_1 and D_2, as shown in Figure 3. Each compartment is filled with the same monoatomic ideal gas with pressure P, volume V, and absolute temperature T as shown in the figure. The mass of partition D_1 is m, which can slide freely without friction, while partition D_2 is fixed and has a small valve on it. Now, the valve on partition D_2 is opened, allowing the gases in compartments B and C to mix and the entire system to reach equilibrium while maintaining a constant temperature T_0.
(a) [3] What are the pressures and volumes of the gases in compartments A, B, and C after the entire system reaches equilibrium?
(b) [3] How much total heat is absorbed by the gases in compartments B and C during the process of the entire system reaching equilibrium?
(c) [4] Calculate the change in entropy ΔS of the entire system during the process of reaching equilibrium.

一封闭的容器以D_1 和 D_2 两隔板分隔成 A、B、和 C 三室，三室中各充以相同的单原子理想气体，其压强 P、体积 V、和绝对温度 T 分别如图三所示。图中 D_1 隔板的质量为 m，可以无摩擦地自由滑动；D_2 隔板固定，其上设有一小活门。现打开 D_2 隔板上的小活门，使 B 和 C 二室中的气体得以混合，并且让整个系统在维持等温 T_0 的情况下达到平衡状态。
(a) [3] 整个系统在达到平衡状态后，A、B、和 C 三室中的气体压强和体积各为何？
(b) [3] 在整个系统达到平衡状态的过程中，B 和 C 两室中的气体合计吸热多少？
(c) [4] 在整个系统达到平衡状态的过程中，计算整个系统的熵的变化量 ΔS。



Solution:
(a) Let the initial number density of gas in compartment A, B and C be n_A, n_B, n_C, After open the have in D_2 and reach the equilibrium, the densities become n'_A, n'_B, n'_C respectively. In the final state, the pressure in all compartments are p_f and temperature T_0. We have

$$p_f V_A = n'_A R T_0$$
$$p_f V'_B = n'_B R T_0$$
$$p_f V'_C = n'_C R T_0$$

Add 3 equations together,

$$p_f(V'_A + V'_B + V'_C) = 3V_0 p_f = (n'_A + n'_B + n_C{}')RT_0 = (n_A + n_B + n_C)RT_0$$

$$\Rightarrow p_f = \frac{1}{3V_0}(n_A + n_B + n_C)RT_0 = \frac{1}{3V_0}(P_0V_0 + P_0V_0 + 2P_0V_0) = \frac{4}{3}P_0$$

For A, since $n'_A = n_A$,

$$p_f V'_A = n'_A R T_0 = n_A R T_0 \Rightarrow V'_A = \frac{3}{4}V_0$$
$$V'_C = V_0$$
$$V'_B = 3V_0 - V_0 - \frac{3}{4}V_0 = \frac{5}{4}V_0$$

(b) (Method 1) Since the entire system is maintained at the same temperature T_0, we have

$$\Delta Q_A + \Delta Q_{BC} = 0$$

For compartment A, since the internal energy doesn't change,

$$\Delta Q_A = \Delta W = \int_{V_0}^{\frac{3}{4}V_0} p_A dV_A = n_A R T_0 \int_{V_0}^{\frac{3}{4}V_0} \frac{1}{V_A} dV_A = P_0 V_0 \ln\frac{3}{4}$$

$$\Rightarrow \Delta Q_{BC} = -\Delta Q_A = P_0 V_0 \ln\frac{4}{3} \approx 0.288\, P_0 V_0$$

(Method 2) Since the gas is maintained at constant temperature T_0, the internal energy of the (ideal) gas doesn't change.

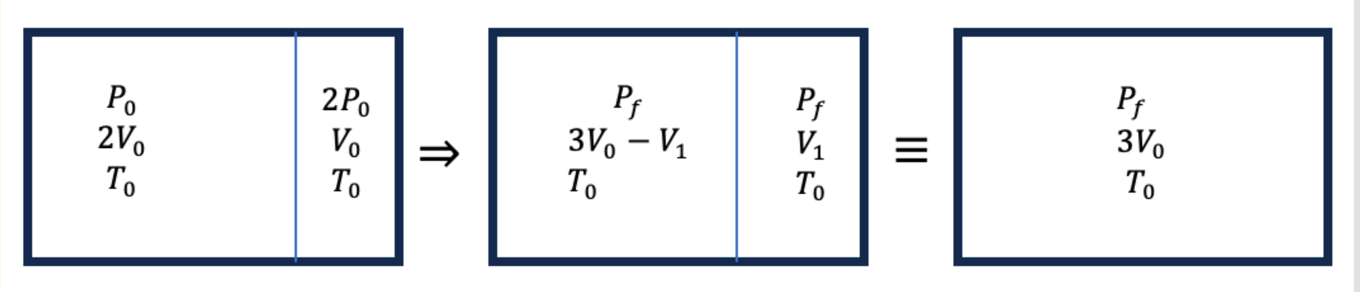
$$\Delta Q = \int_{V_0}^{\frac{5}{4}V_0} p_B dV_B = \int_{V_0}^{\frac{5}{4}V_0} p_A dV_B = \int_{V_0}^{\frac{5}{4}V_0} \frac{N_A k T_0}{2V_0 - V_B} dV_B = N_A k T_0 \ln\frac{4}{3} = P_0 V_0 \ln\frac{4}{3} \approx 0.288\, P_0 V_0$$

(c) Notice that the gas flow between compartment B and C is irreversible therefore we can't simply apply the formula

$$dS = \frac{dQ}{T}$$

(Method 1) One way we can do is to construct a reversible path which connect the initial state to the final state. We imagine there is a partition between A+B and C which can move slowly. The gas in C can expand so that the pressure decreases from $2P_0$ to P_f. We would like to determine V_1 such that the

$$2P_0V_0 = \frac{4}{3}P_0V_1 \Rightarrow V_1 = \frac{3}{2}V_0$$



We can imagine there is a partition between A+B and C and move slowly,

$$\Delta S = \frac{dQ}{T_0} = \frac{pdV}{T_0} = \frac{(p_C - p_{AB})dV_C}{T_0}$$

$$\Rightarrow \Delta S = \int_{V_0}^{\frac{3}{2}V_0} \left(\frac{2Nk}{V_C} - \frac{2Nk}{3V_0 - V_C}\right) dV_C = 2Nk \int_{V_0}^{\frac{3}{2}V_0} \left(\frac{1}{V_C} - \frac{1}{3V_0 - V_C}\right) dV_C = 2Nk\left(\ln\frac{3}{2} + \ln\frac{3}{4}\right) = 2Nk \ln\frac{9}{8} = 2\frac{P_0V_0}{T_0}\ln\frac{9}{8}$$
$$\approx 0.236\frac{P_0V_0}{T_0}$$

(Method 2) To get the entropy of BC, we apply the Sackur-Tetrode equation for the ideal gas,

$$S = Nk\left[\ln\left(\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right] = Nk\left[\ln\left(\frac{V}{N}\left(\frac{2\pi mkT_0}{h^2}\right)^{3/2}\right) + \frac{5}{2}\right] = Nk\left(\ln\frac{VT_0^{\frac{3}{2}}}{N} + c\right) = -Nk\ln P + C$$

when T_0 is constant.

Initial entropy

$$S_i = -2\frac{P_0V_0}{T_0}\ln P_0 - \frac{2P_0V_0}{T_0}\ln 2P_0$$

Final state:

$$S_f = -4Nk\ln P_f = -4Nk\ln\frac{4}{3}P_0$$

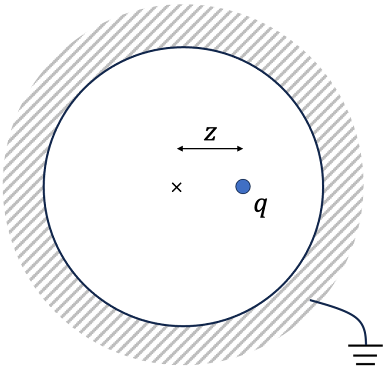
$$\Delta S = S_f - S_i = -\frac{4P_0V_0}{T_0}\ln\frac{4}{3}P_0 + 2\frac{P_0V_0}{T_0}\ln P_0 + \frac{2P_0V_0}{T_0}\ln 2P_0 = \frac{P_0V_0}{T_0}\left(-4\ln\frac{4}{3}P_0 + 2\ln P_0 + 2\ln 2P_0\right)$$
$$= \frac{P_0V_0}{T_0}\left(-4\ln\frac{4}{3} + 2\ln 2\right) = 2\frac{P_0V_0}{T_0}\left(\ln\frac{9}{8}\right) \approx 0.236\frac{P_0V_0}{T_0}$$

3. [10 points] Trapped Ball
A ball (modelled as a point charge of magnitude $q > 0$) of mass m is trapped in a spherical cavity of radius R carved from an infinite grounded conductor. The charge is at a distance z from the center. In the problem, the gravity can be ignored.

(a) [2] Sketch the electric field lines inside the spherical cavity.
(b) [4] Find the electric force $F(z)$ acting on the ball in terms of q, z and R.
(c) [4] If the ball is released at the center with a very small speed, find the speed of the ball v when it is at a distance $R/2$ from the center of the conductor. Express the answer in terms of q, m, R.

3. [10 分] 困在球内的小球
一颗质量为 m 的小球（假设为一个大小为 $q > 0$ 的点电荷）被困在一个由无限大的接地导体挖空而成的半径为 R 的球形空腔中。该电荷距离中心的距离为z。在这个问题中，可以忽略重力。

(a) [2] 描绘球形空腔内的电场线。
(b) [4] 求作用于小球的电力 $F(z)$，用 q ，z和 R 表示答案。
(c) [4] 如果小球在中心以极小的速度释放，求当小球距导体中心距离为$R/2$ 时的速度v。用 q ，m ，R 表示答案。



Solution:
(a) Inside the conductor, the electric field is zero. Electric field lines should terminate perpendicular to the conductor. Due to accumulation of surface charges on the wall of the cavity, the electric field strength on the 'right hand side' of the charge should be stronger than that on the 'left hand side' and hence the density of electric field lines should be higher. Finally, note that as the charge is positive, the electric field lines should point out from the charge. With these considerations in mind, we have the electric field lines shown in Fig. (a):

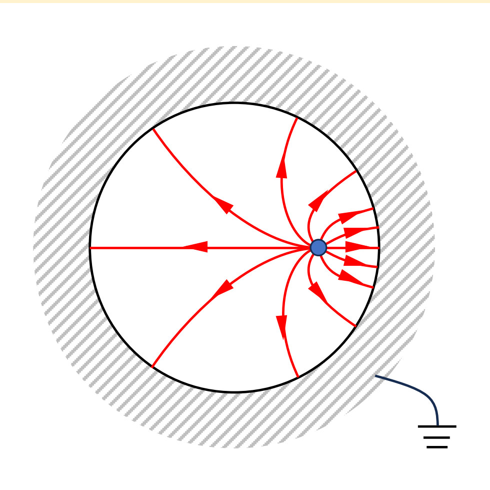


Figure (a)

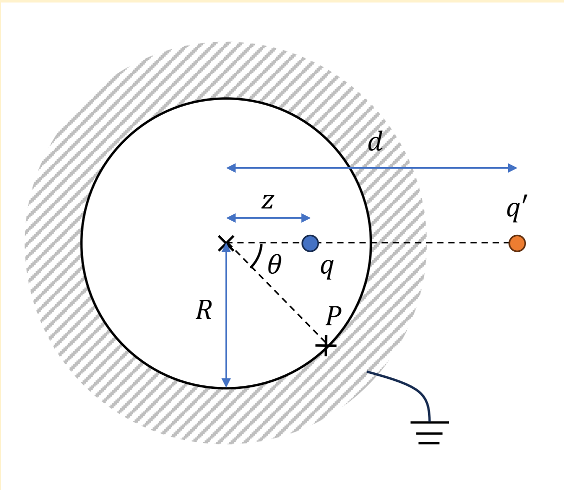


Figure (b)

(b) To solve this problem, we apply the method of image charges, with an image charge of magnitude $q' < 0$ located at d from the center of the spherical cavity as shown in Fig. (b). Consider the potential $\phi_P(\theta)$ at a point P on the wall:

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{z^2+R^2-2zR\cos\theta}}+\frac{q'}{\sqrt{d^2+R^2-2dR\cos\theta}}\right)=\phi_P(\theta)=0$$

$$\Rightarrow \frac{q^2}{z^2+R^2-2zR\cos\theta}=\frac{{q'}^2}{d^2+R^2-2dR\cos\theta}$$

$$\Rightarrow q^2(d^2+R^2)-{q'}^2(z^2+R^2)+2R({q'}^2z-dq^2)\cos\theta=0$$

For this to be satisfied by all θ, we need

$$\begin{cases} {q'}^2z-dq^2=0 \\ q^2(d^2+R^2)-{q'}^2(z^2+R^2)=0 \end{cases}$$

Solving gives

$$d=\frac{R^2}{z} \quad \text{and} \quad q'=-\frac{qR}{z}$$

This is the almost the same work we did when we solved the image charge problem for a charge outside a grounded conducting sphere. This gives our force (attractive) to be

$$|F|=\frac{1}{4\pi\varepsilon_0}\frac{|qq'|}{(d-z)^2}=\frac{1}{4\pi\varepsilon_0}\frac{q^2Rz}{(R^2-z^2)^2}$$

(ci) By the work-energy theorem, the velocity at $z=R$ is given by

$$\frac{1}{2}m(v^2-0^2)=\int_0^{R/2}F(z)\,\mathrm{d}z$$

$$v=\sqrt{\frac{2}{m}\int_0^{R/2}F(z)\,\mathrm{d}z}=\sqrt{\frac{q^2R}{2\pi\epsilon_0 m}\int_0^{R/2}\frac{z}{(R^2-z^2)^2}dz}=\sqrt{\frac{q^2}{2\pi\epsilon_0 mR}\int_0^{1/2}\frac{u}{(1-u^2)^2}du}=\sqrt{\frac{1}{12\pi\varepsilon_0}\frac{q^2}{mR}}$$

4. [10 points] <u>In this question, all answers cannot be expressed in terms of any trigonometrical functions.</u>

An ice hemisphere with radius R and refractive index n lies on a warm flat table and melts slowly. The rate of heat transfer from the table to the ice is proportional to the area of contact between them. It is known that the ice hemisphere completely melts in time T_0. Throughout the process, a laser beam incident on the ice from above. The beam is vertically incident at a distance of $R/2$ from the axis of symmetry (see figure).

Assume that the temperature of the ice and the surrounding atmosphere are 0°C and remains constant during the melting process. The laser beam does not transfer energy to the ice. The melting water immediately flows off the table, and the ice does not move along the table.

(a) [4] What is the position of the point on the table, $x_0=x(t=0)$, where the beam hit at time $t=0$? Express the answer in terms of n and R.

(b) [3] What is the height of the ice $z(t)$ as a function of time t? Express the answer in terms of R and T_0.

(c) [3] What is the position of the point on the table, $x(t)$, where the beam hit for $t\geq 0$? Express the answer in terms of n, R, T_0 and t.

4. [10 分] <u>在这个问题中，所有答案都不能用任何三角函数来表达。</u>

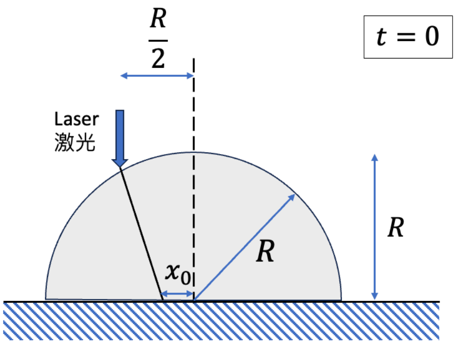
一个半径 R 和折射率 n 的冰半球位于一个温暖的平面桌子上，并缓慢融化。桌子和冰之间的传热速率与它们的接触面积成正比。已知冰半球在时间 T_0 内完全融化。在整个过程中，激光束从上方照射到冰上。光束从垂直于对称轴距离为 $R/2$ 的位置入射（见图）。

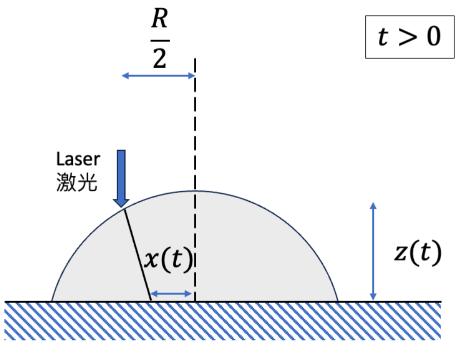
假设冰和周围大气的温度都是$0°C$，并且在融化过程中保持不变。激光束不会向冰传递能量。溶化了的水立即流到桌子外，冰不会沿桌子移动。

(a) [4] 当$t = 0$时，光束击中桌子上的点 $x_0=x(t=0)$ 的位置在哪里？用 n 和 R 表示答案。

(b) [3] 冰的高度 $z(t)$ 作为时间 t 的函数是多少？用 R 和 T_0 表示答案。

(c) [3] 对于 $t\geq 0$，光束击中桌子上的点 $x(t)$ 的位置在哪里？用n ，R ，T_0 和 t 表示答案。





Solution:

(a) Let's find the point on the table where the beam hits at the very first moment $t = 0$. Consider Figure 5, it is easy to find that the angle of incidence of the beam on the hemisphere is $\alpha = 30°$, i.e. $\sin\alpha = \frac{1}{2}$.

According to Snell's law of refraction

$$\sin\beta = \frac{\sin\alpha}{n} = \frac{1}{2n} \Rightarrow \tan\beta = \frac{1}{\sqrt{4n^2-1}}$$

From the pink triangle with angle γ in Figure 5 it is easy to find its horizontal leg $\frac{\sqrt{3}R}{2}\tan\gamma$, which means the point of incidence of the beam on the table: it is located at a distance x_0 from the center of hemisphere O:

$$x_0 = \frac{R}{2} - \frac{\sqrt{3}}{2}R\tan\gamma$$

$$\tan\gamma = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\sqrt{4n^2-1}-\sqrt{3}}{\sqrt{12n^2-3}+1}$$

$$x_0 = \frac{R}{2}\left(1 - \frac{\sqrt{12n^2-3}-3}{\sqrt{12n^2-3}+1}\right) = \frac{2R}{\sqrt{12n^2-3}+1}$$

(b) At time t, the base in contact with the table is a circle with radius r. Within a time interval dt, the amount of heat absorbed is

$$\mathrm{d}Q = K\pi r^2 dt$$

for some constant K. The volume of the ice melt is

$$dQ = Ldm = L\rho\pi r^2 dz$$

Where L and ρ are the latent heat and density of the ice respectively. Therefore we have

$$K\pi r^2 dt = -L\rho\pi r^2 dz$$

$$\Rightarrow \frac{dz}{dt} = -\frac{K}{L\rho} \text{ is a constant}$$

i.e. the melting uniformly reduces the thickness of the ice, we have

$$z(t) = R\left(1 - \frac{t}{T_0}\right)$$

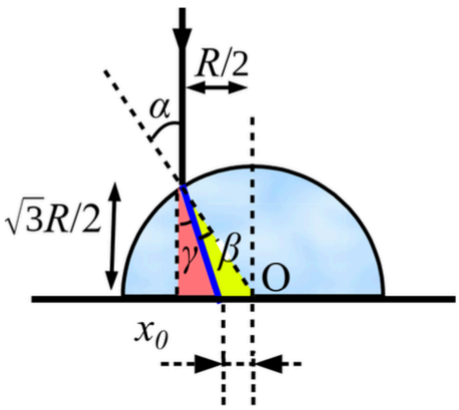
(c) At the end, when the base of the ice has radius $r \le \frac{R}{2}$, the beam will obviously hit the table vertically (i.e. $x(t) = \frac{R}{2}$). It corresponds to the case when the thickness of the ice reduces to $z(t) = \left(1 - \frac{\sqrt{3}}{2}\right)R$. It will happen at the time $t = \frac{\sqrt{3}}{2}T_0$. Therefore, for $t > \frac{\sqrt{3}}{2}T_0$, we have $x(t) = \frac{R}{2}$

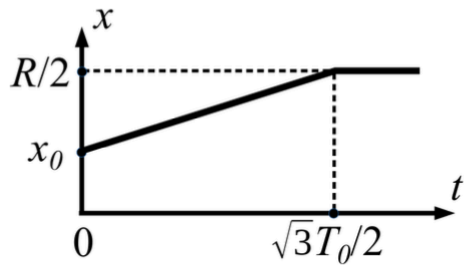
To get $x(t)$ for $t < \frac{\sqrt{3}}{2}T_0$, we can consider the similar triangles.

$$\frac{\frac{R}{2} - x_0}{\frac{R}{2} - x(t)} = \frac{\frac{\sqrt{3}}{2}R}{R\left(\frac{\sqrt{3}}{2} - \frac{t}{T_0}\right)}$$

$$\Rightarrow \left(\frac{R}{2} - x_0\right)\left(\frac{\sqrt{3}}{2} - \frac{t}{T_0}\right) = \frac{\sqrt{3}}{2}\left(\frac{R}{2} - x\right)$$

$$\Rightarrow x(t) = \frac{R}{2} - \left(\frac{R}{2} - x_0\right)\left(1 - \frac{2}{\sqrt{3}}\frac{t}{T_0}\right) = \frac{R}{2} - \frac{R}{2}\left(1 - \frac{4}{\sqrt{12n^2-3}+1}\right)\left(1 - \frac{2}{\sqrt{3}}\frac{t}{T_0}\right)$$





~ End of Part 1 卷-1 完 ~