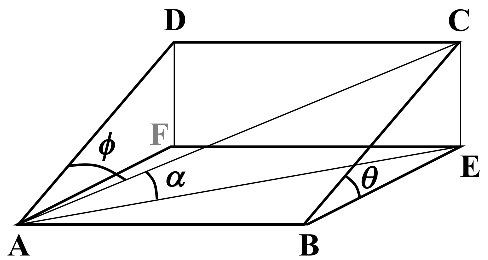
Pan Pearl River Delta Physics Olympiad 2018
2018 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 6 Problems, 45 Points) 简体版卷-1（共6题，45分）
(9:00 am – 11:45 am, 22 February, 2018)

Please fill in your final answers to all problems on the **answer sheet**.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the **answer sheet only**. Question papers and working sheets will **not** be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. Length of Daytime (7 points) 白昼长度（7 分）

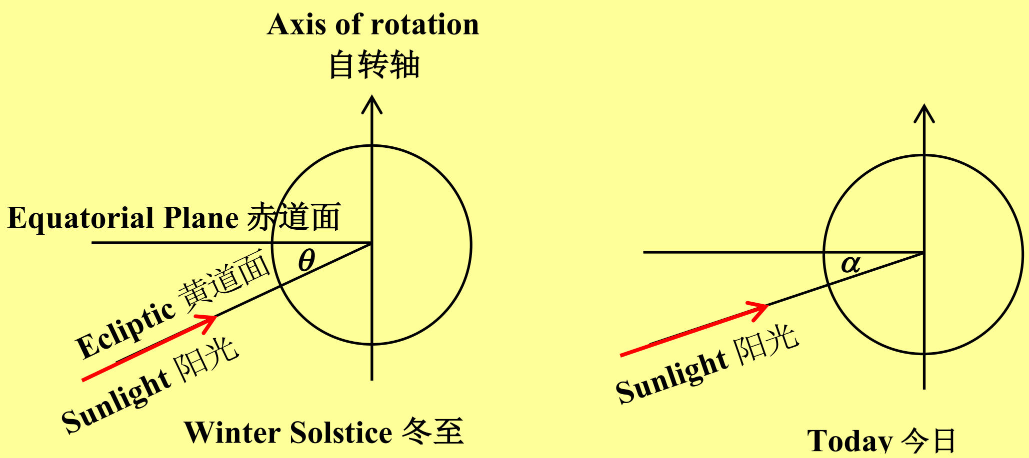
(a) In the figure, ABCD is a rectangle lying on an inclined plane making an angle θ with the horizontal plane. ABEF is the projection of the rectangle on the horizontal plane. If the measure of the angle DAC is ϕ, derive an expression for the angle α. [2]
如图所示，矩形 ABCD 位于斜面上，斜面与水平面夹角为θ。ABEF 为该矩形于水平面的投影。设角 DAC 为ϕ，试推导角α的表达式。[2]



Let h = AC. Then
CE $= h\sin\alpha$.
BC $=$ AD $= h\cos\phi$.
CE $=$ BC $\sin\theta = h\cos\phi\sin\theta$.
Equating the expressions of CE, $h\sin\alpha = h\cos\phi\sin\theta \Rightarrow \alpha = \arcsin(\cos\phi\sin\theta)$.

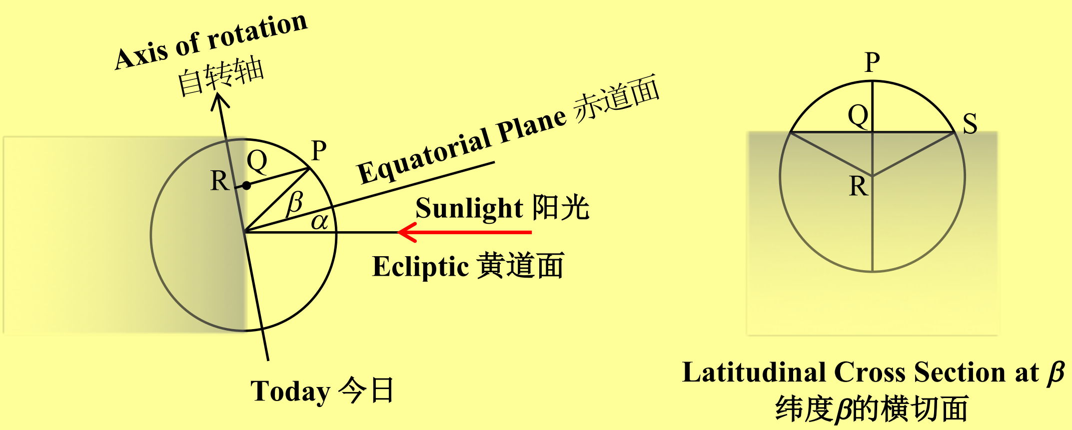
(b) The ecliptic is the plane on which the Earth revolves around the Sun. The axis of rotation of the Earth is inclined at an angle of 23.4^{o} with the normal to the ecliptic. The day of the Winter Solstice (in the Northern Hemisphere) is 21 December. Using the result of (a) or otherwise, calculate the incident angle of sunlight relative to Earth's equatorial plane today (22 February). [2]
黄道面是指地球围绕太阳公转的平面。地球的自转轴相对于黄道面的法线倾斜，角度为 23.4^{o}。在北半球，冬至的日期为 12 月 21 日。试用(a)部结果或其他方法，计算今天（2 月 22 日）阳光相对于赤道面的角度。[2]

In the figure above, consider ABEF to be the equatorial plane of the Earth, and ABCD the ecliptic. Then θ = 23.4^{o}. When the Earth revolves around the Sun, sunlight is incident on the Earth from different directions lying on the plane ABCD. For example, on 21 December, sunlight is incident on the Earth in the direction AD, since this is the southernmost direction of sunlight. Similarly, during Spring Equinox and Autumn Equinox, sunlight is incident on the Earth in the direction AB or BA.



On 22 February, the time is 63 days after the Winter Solstice. Hence on the ecliptic, sunlight is incident from the angle $\phi = 360^0\left(\frac{63}{365}\right) = 62.14^0$, and relative to the equatorial plane, it is incident from the angle
$\alpha = \arcsin(\cos\phi\sin\theta) = \arcsin[\cos(62.14^0)\sin(23.4^0)] = 10.70^0$.

(c) The latitude of Hong Kong is β = 22.25°. Calculate the length of daytime in Hong Kong today (22 February). Give your answer in hours to 3 significant figures. [3]
香港位于北纬β = 22.25°。试计算今天(2 月 22 日) 香港白昼的长度。答案请以小时表达，给三位有效数字。[3]



Consider the latitudinal cross section at β.
PR $= R\cos\beta$. (R is the radius of the Earth.)
QR $= R\sin\beta\tan\alpha$.
Hence the angle QRS is given by $\cos x = \frac{\mathrm{QR}}{\mathrm{SR}} = \frac{R\sin\beta\tan\alpha}{R\cos\beta} = \tan\beta\tan\alpha$.
$x = \arccos(\tan\beta\tan\alpha) = \arccos(\tan 22.25^0 \tan 62.14^0) = 85.57^0$.
Length of daytime in Hong Kong $= 24\left(\frac{85.57}{180}\right) = 11.4$ h

2. Rotating Ball (6 points) 滚動的球（9 分）

A ball of mass m and radius a is at rest on the surface of a sphere with radius R. The ball is initially at the angle θ_0 at $t = 0$. The bottom sphere is fixed and cannot move, but there is friction so the ball rolls without slipping until it leaves the surface of the sphere at angle θ_1.

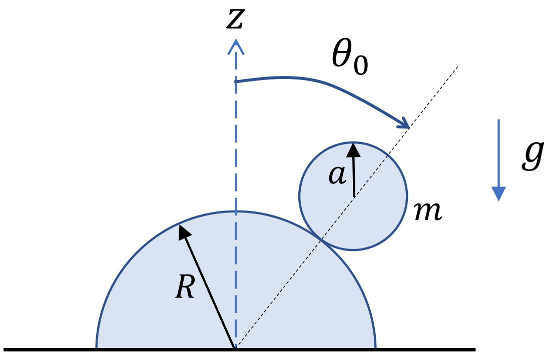
The moment of inertia of the ball about its axis is $I = \frac{2}{5}ma^2$. The gravity is directed as shown.
一个质量 m 和半径 a 的球放在一个半径 R 的球体表面上。球最初处于角度θ_0处。底部球体是固定的并且不能移动，但是存在摩擦力，因此球作纯滚动，直至它以角度θ_1离开球体表面。 球围绕其轴线的惯性矩是 $I = \frac{2}{5}ma^2$。引力方向如图所示。

(a) What is the frictional force acting on the ball at angle θ? [3]
在角度θ时作用在球上的摩擦力是多少？[3]

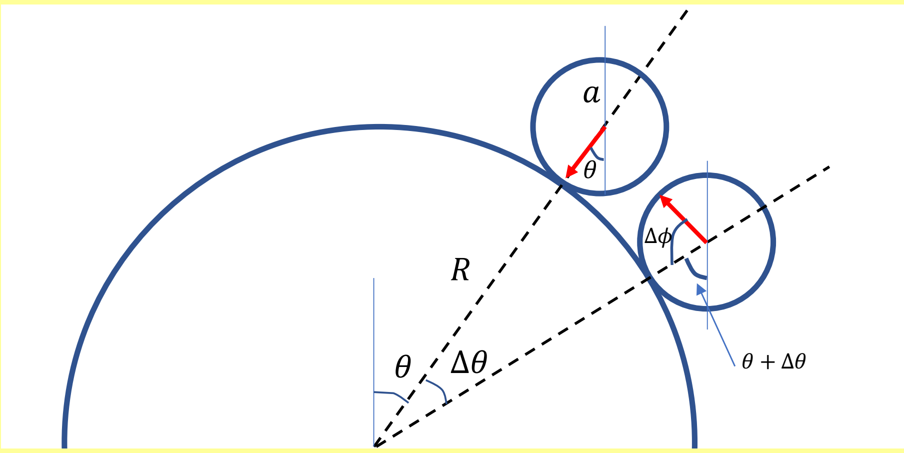
(b) What is the normal force acting on the ball at angle θ? [3]
在角度θ时作用在球上的支持力是多少？[3]

(c) Find θ_1 in terms of θ_0. [1]
试推导θ_1的表达式，用θ_0表示。[1]

(d) What is the speed of the ball at angle θ_1? [2]
在角度θ_1時球的速率是多少？[2]



Solution:
(a)



If the ball is rolling without slippering,

$$R\Delta\theta = a\Delta\phi$$

$$\omega \equiv \frac{(\theta + \Delta\theta + \Delta\phi) - \theta}{\Delta t} = \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{d\theta}{dt}\left(\frac{a+R}{a}\right)$$

$$(R + a)\Delta\theta = v\Delta t$$

$$\rightarrow v = (R+a)\frac{\Delta\theta}{\Delta t} = a\omega$$

By the conservation of energy and Newton's 2nd law, we have

$$mg(R+a)(1+\cos\theta_0) - mg(R+a)(1+\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$\rightarrow mg(R+a)(\cos\theta_0 - \cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}ma^2\frac{v^2}{\mathrm{a}^2} = \frac{7}{10}mv^2$$

$$\rightarrow v^2 = \frac{10}{7}g(R+a)(\cos\theta_0 - \cos\theta) \qquad (a)$$

From Newton's 2nd law on rotation

$$fa = I\frac{d\omega}{dt} \rightarrow f = \frac{2}{5}ma\frac{d\omega}{dt}$$

Newton's 2nd law along the tangential motion,

$$mg\sin\theta - f = m\frac{dv}{dt} = ma\frac{d\omega}{dt} = \frac{5}{2}f$$

$$\rightarrow f = \frac{2}{7}mg\sin\theta$$

Another way to get the result is from equation (a) without using the Newton's 2nd law. Since $v = (R+a)\frac{d\theta}{dt}$, *Eq. (a) implies*

$$(R+a)^2\left(\frac{d\theta}{dt}\right)^2 = \frac{10}{7}g(R+a)(\cos\theta_0 - \cos\theta)$$

Differentiate on both sides,

$$2(R+a)^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = \frac{10}{7}g(R+a)\sin\theta\frac{d\theta}{dt} \rightarrow \frac{d^2\theta}{dt^2} = \frac{5}{7}\frac{g}{R+a}\sin\theta$$

And

$$f = \frac{2}{5}ma\frac{d\omega}{dt} = \frac{2}{5}m(a+R)\frac{d^2\theta}{dt^2} = \frac{2}{7}mg\sin\theta$$

(b) Newton's 2nd law gives

$$mg\cos\theta - N = \frac{mv^2}{R+a}$$

The normal force is

$$N = mg\cos\theta - \frac{mv^2}{R+a} = mg\cos\theta - \frac{m}{R+a}\frac{10}{7}g(R+a)(\cos\theta_0 - \cos\theta)$$

$$= mg(\frac{17}{7}\cos\theta - \frac{10}{7}\cos\theta_0)$$

(c) At θ_1, the normal force vanishes, we have

$$\cos\theta_1 = \frac{10}{17}\cos\theta$$

(d) the velocity of the ball at the angle θ is

$$v = \sqrt{g(R+a)(\cos\theta_0 - \cos\theta_1)} = \sqrt{\frac{7}{17}g(R+a)\cos\theta_0}$$

3. Nearest Exoplanet Discovered (7 points) 发现最近的系外行星（7 分）

On 24th August 2016, astronomers discovered a planet orbiting the closest star to the Sun, Proxima Centauri, situated 4.22 light years away, which fulfils a long-standing dream of science-fiction writers: a world that is close enough for humans to send their first interstellar spacecraft.
2016 年 8 月 24 日，天文学家发现在距离太阳最近的恒星一比邻星（Proxima Centauri），有一颗行星围绕着它运行。比邻星距离太阳 4.22 光年。这发现实现了科幻小说作家的长期梦想：一个足够接近的世界，人类可以把第一艘星际航天器送达。

Astronomers have noted how the motion of Proxima Centauri changed in the first months of 2016, with the star moving towards and away from the Earth. In the figure below, the radial velocities of the star are measured and the direction of the radial velocities changed regularly. This regular pattern caused by an unseen planet, which they named Proxima Centauri B, repeats and results in tiny Doppler shifts in the star's light, making the light appear slightly redder, then bluer.
天文学家注意到比邻星的运动在 2016 年的头几个月的变化。恒星规律性地朝着和远离地球移动 。在下图中， 透过测量恒星的径向速度，發現径向速度的方向有规律地变化。径向速度的规律性变化是由一颗看不见的行星引起的，该行星称为比邻星 B，会重复导致恒星光线发生微小的多普勒频移，从而使光线稍微变红，然后变蓝。

It it is given that the star, Proxima Centauri, has a surface temperature of 3000 K and a radius of $R = 0.14R_{Sun}$ and the orbit of the unseen planet, Proxima Centauri B, around the star is circular. (Radius of the Sun $R_{Sun} = 6.96\times10^{8}\text{m}$, the gravitational constant $G = 6.674\times10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2$)
恒星比邻星的表面温度为3000 K，半径 $R = 0.14R_{Sun}$，行星比邻星 B 围绕恒星的轨道是圆形。（太阳半径$R_{Sun} = 6.96\times10^{8}\text{m}$，引力常数$G = 6.674\times10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2$）

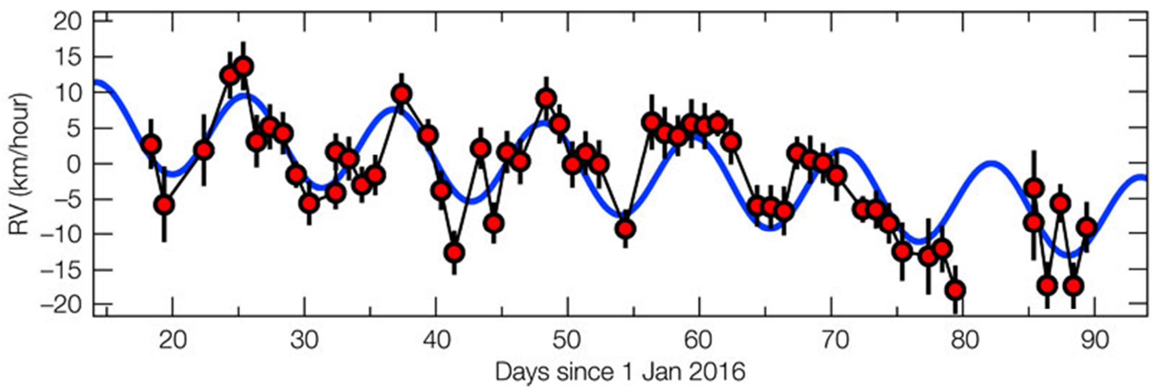


Figure: Measurements of the radial velocity of Proxima Centauri (denoted as RV in the figure) from 1 January 2016 onwards. The blue curve is the best fit curve of the data.
图：从 2016 年 1 月 1 日起，比邻星的径向速度（在图中以 RV 表示）测量结果。 蓝色曲线是数据的最佳拟合曲线。

(a) Proxima Centauri is a red dwarf star, unlike our Sun, with a mass of only 0.12 M_{Sun}. Estimate the radius of the planet's orbit using the given information. (The mass of sun, $M_{Sun} = 1.989\times10^{30}\text{kg}$) [2]
比邻星是一颗红矮星， 与我们的太阳不同， 质量仅为$0.12M_{Sun}$。试用所给资料估算行星轨道的半径。(太阳的质量 $M_{Sun} = 1.989\times10^{30}\text{kg}$) [2]

(b) Estimate the mass of the planet in terms of Earth mass. ($M_{Earth} = 5.972\times10^{24}kg$) [2]
试估算行星的质量，以地球质量为单位表示。（地球质量 $M_{Earth} = 5.972\times10^{24}\mathrm{kg}$））[2]

(c) Estimate the equilibrium temperature of the planet by assuming that both the star and planet are black bodies. [3]
假设恒星和行星都是黑体，试估算行星的稳态温度。[3]

Solution:
(a) The period is the time interval between two consecutive peaks of the curve. From the radial velocity curve, the period is 11 days
(Full marks for the period within ±2 days)
Using Kepler's 3rd law: (Assuming $M_{star} \gg M_{planet}$)

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_{star}} \to a = 7.14\times10^9\mathrm{m}$$

(b) The orbital velocity for a circular orbit is:

$$v = \frac{2\pi a}{T} = 47200\mathrm{m/s}$$

Here a is the distance between the star and the planet, that is, the sum of the orbital radii of the planet and the star, and we assume that the orbital radius of the star is negligible.
In the center of mass frame, $\vec{p}_{cm} = 0$.

$$\to Mv_s - mv_p = 0$$

From the figure, $v_s = 6.5\ \mathrm{km/h} = 1.806\ \mathrm{m/s}$.

$$\to m = \frac{v_s}{v_p}M = \left(\frac{1.806}{47200}\right)(0.12\times1.989\times10^{30}) = 9.1\times10^{24}kg \approx 1.5M_{Earth}$$

(c) By assuming thermal equilibrium, the heat absorbed by the planet (due to the radiation of star) is equal to the heat radiated by the planet (as a blackbody)

$$\left(\frac{4\pi R_s^2\sigma T_s^4}{4\pi a^2}\right)\pi R_p^2 = 4\pi R_p^2\sigma T_p^4 \to T_p^4 = \frac{R_s^2}{4a^2}T_s^4 \to T_p = T_s\sqrt{\frac{R_s}{2a}} \approx 248\mathrm{K} = -25°C$$

4. L-Shaped Conductor with a Wire (8 points) L 形导体和导线（8 分）

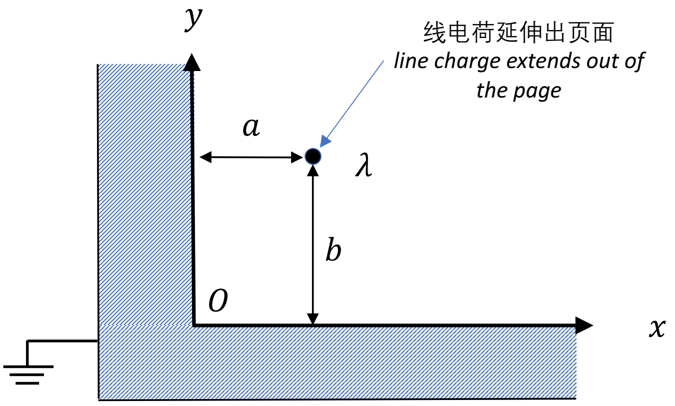
A L-shaped conductor consists of two semi-infinite conductors in the *xz* and *yz* planes where the cross section is shown in the figure. The L-shaped conductor is grounded and centered at the origin. A line of charge, with linear charge density λ runs parallel to the *z*-axis is located at (a, b) where $b > a > 0$.
L 形导体由 *xz* 和 *yz* 平面中的两个半无限平面组成，图中显示了导体的横截面。 L 形导体接地，中心点为原点。一条线性电荷密度为λ与 *z* 轴平行的电荷线位于（*a*,*b*)，其中 $b > a > 0$。

(a) Compute the electric potential $V(x, y, z)$ for $x > 0$ and $y > 0$. [3]
计算$x > 0$和$y > 0$时的电势 $V(x, y, z)$。[3]

(b) Compute the capacitance per unit length of a thin wire of radius r, placed at the point (a, b). Assume that the wire radius is much smaller than a and b (i.e. $r \ll a, b$) so that the solution of part (a) is approximately correct in the region exclusive of the conductors. [3]

计算放置在点(a,b)处、半径为 r 的细导线，其每单位长度的电容。 假定线半径 r 比 a 和 b 小得多（即$r \ll a,b$ ）， 使得在(a)部的解在除导体之外的区域中近似正确。[3]

(c) Compute the force per unit length on the wire (as a vector). [2]
计算导线上每单位长度的力（作为矢量）。[2]



Solution:
(a) We apply image method by adding 3 image line charge in the following ways:

1. $(-a,b)$: Charge density $-\lambda$
2. $(-a,-b)$: Charge density λ
3. $(a,-b)$: Charge density $-\lambda$

Hence the total electric field along the x-axis (at the point $(x,0)$) is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{(x-a)^2+b^2}\left((x-a)\hat{\imath}-b\hat{\jmath}\right)-\frac{1}{(x+a)^2+b^2}\left((x+a)\hat{\imath}-b\hat{\jmath}\right)\right.$$
$$\left.+\frac{1}{(x+a)^2+b^2}\left((x+a)\hat{\imath}+b\hat{\jmath}\right)-\frac{1}{(x-a)^2+b^2}\left((x-\mathrm{a})\hat{\imath}+b\hat{\jmath}\right)\right)$$
$$=\frac{\lambda}{2\pi\epsilon_0}\left(-\frac{2b}{(x-a)^2+b^2}+\frac{2b}{(x+a)^2+b^2}\right)\hat{\jmath}$$

which is along the y-direction and hence the electric potential which is constant along the x-axis.
Similarly, we can show the electric potential is also constant along the y-axis.
By setting the potential $V(0)=0$ at the origin, the electric potential of an infinite line of charge at (a,b) is

$$V_0(x,y,z)=\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x-a)^2+(y-b)^2}\right)$$

Similarly, we can get the electric potential for other image wires

$$V_1(x,y,z)=-\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x+a)^2+(y-b)^2}\right)$$
$$V_2(x,y,z)=\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x+a)^2+(y+b)^2}\right)$$
$$V_3(x,y,z)=-\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+b^2}{(x-a)^2+(y+b)^2}\right)$$

The total electric potential becomes,

$$V(x,y,z)=\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{(x+a)^2+(y-b)^2}{(x-a)^2+(y-b)^2}\right)\left(\frac{(x-a)^2+(y+b)^2}{(x+a)^2+(y+b)^2}\right)$$

(b)

$$C = \frac{Q}{\Delta V} = \frac{Q}{V(a-r,b,0) - V(0,0,0)} = \frac{\lambda L}{\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{(2a-r)^2}{(-r)^2}\right)\left(\frac{(r)^2 + (2b)^2}{(2a-r)^2 + (2b)^2}\right)}$$

$$= \frac{4\pi\epsilon_0 L}{\ln\left(\frac{(2ab)^2}{(a^2+b^2)r^2}\right)} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{2ab}{r\sqrt{a^2+b^2}}\right)}$$

$$\rightarrow \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{2ab}{r\sqrt{a^2+b^2}}\right)}$$

(c) The force on the wire is the electric force on the wire from three image wires.
From Gauss's law, the electric field created by three image wires at the point $(x, y, 0)$,

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\left(-\frac{1}{(x+a)^2 + (y-b)^2}\left((x+a)\hat{\imath} + (y-b)\hat{\jmath}\right) + \frac{1}{(x+a)^2 + (y+b)^2}\left((x+a)\hat{\imath} + (y+b)\hat{\jmath}\right) - \frac{1}{(x-a)^2 + (y+b)^2}\left((x-a)\hat{\imath} + (y+b)\hat{\jmath}\right)\right)$$

Substitute $(x, y) = (a, b)$, we get

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{(a\hat{\imath} + b\hat{\jmath})}{a^2+b^2} - \frac{1}{a}\hat{\imath} - \frac{1}{b}\hat{\jmath}\right)$$

The force on the wire is

$$\vec{F} = q\vec{E} = \lambda L \frac{\lambda}{4\pi\epsilon_0}\left(\frac{(a\hat{\imath} + b\hat{\jmath})}{a^2+b^2} - \frac{1}{a}\hat{\imath} - \frac{1}{b}\hat{\jmath}\right) \rightarrow \frac{\vec{F}}{L} = \frac{\lambda^2}{4\pi\epsilon_0}\left(\frac{(a\hat{\imath} + b\hat{\jmath})}{a^2+b^2} - \frac{1}{a}\hat{\imath} - \frac{1}{b}\hat{\jmath}\right)$$

5. A Flying Square Loop (8 points) 一個飛行的方形環(8 分)

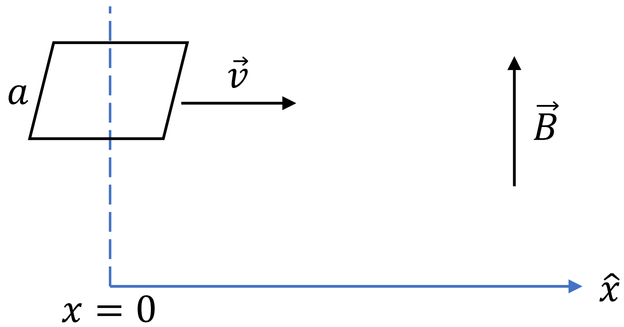
A square loop of side a and mass m is made of resistive material with a total resistance R. At $t = 0$, the loop is located at $x = 0$ and moves with velocity $v_0\hat{x}$. The loop lies in the x-y plane. There is a magnetic field $\vec{B} = B_0\left(\frac{x}{x_0}\right)\hat{z}$ where $B_0 > 0$ is a constant. In the problem, we neglect the effect of gravity.
一边长为 a、质量为 m 的方形环由电阻材料制成，总电阻为R。在$t = 0$时，环位于$x = 0$并以速度$v_0\hat{x}$ 移动。 环位于 x-y 平面中。 有一磁场 $\vec{B} = B_0\left(\frac{x}{x_0}\right)\hat{z}$，其中$B_0 > 0$是一常数。在这问题中，我们忽略重力的影响。

(a) What is the induced current on the loop when the center of the loop is at the point x with velocity $v\hat{x}$? What is the direction of the current? Is it clockwise/anticlockwise from above? [2]
当方形环中心位于x、速度为$v\hat{x}$ 时，环上的感应电流是多少？ 方向是什么？ 从上方观看是顺时针/逆时针？[2]

(b) What is the velocity of the square loop $v(t)$ at time t? [3]
方形环在时间t的速度$v(t)$ 是多少？[3]

(c) How far does the loop travel before stopping? [3]

方形环在停止前的行进距离是多少？[3]



Solution:

(a) When the center of the square loop is at the point x, the magnetic flux passes through it is,

$$\Phi = a\int_{x=x-\frac{a}{2}}^{x+\frac{a}{2}} B_0\left(\frac{x}{x_0}\right)dx = \frac{aB_0}{2x_0}\left(\left(x+\frac{a}{2}\right)^2 - \left(x-\frac{a}{2}\right)^2\right) = \frac{aB_0}{2x_0}(2ax) = \frac{a^2B_0x}{x_0}$$

The induced emf is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{a^2B_0}{x_0}\left(\frac{dx}{dt}\right) = -\frac{a^2B_0}{x_0}v$$

The induced current is

$$I = \frac{\mathcal{E}}{R} = \frac{a^2B_0}{x_0R}v$$

and the current is clockwise from above.

(b) The dissipation power of the induced current compensated by the work done to decelerate the loop,

$$P = I\mathcal{E} = \frac{\mathcal{E}^2}{R} = \frac{1}{R}\left(\frac{a^2B_0}{x_0}\right)^2 v^2 = -m\dot{v}v$$

$$\rightarrow \frac{\dot{v}}{v} = -\frac{1}{mR}\left(\frac{a^2B_0}{x_0}\right)^2 = -\frac{\gamma}{m} \qquad where\ \gamma = \frac{1}{R}\left(\frac{a^2B_0}{x_0}\right)^2$$

The velocity of the loop is

$$v(t) = v_0e^{-\frac{\gamma}{m}t}$$

(c) Total distance travelled is

$$x = \int_{t=0}^{\infty} v(t)dt = \frac{v_0m}{\gamma} = Rmv_0\left(\frac{x_0}{a^2B_0}\right)^2$$

6. The Phenomenon of the Halo (6 Points) 光晕现象（6 分）

Bright halos around the sun can be observed as in Figure 1. As shown in Fig. 2, this optical phenomenon is caused by the refraction of the sun's rays on ice crystals in the cirrostratus, a cloud genus that reaches a height of approximately 5.5 km.

我们有时候可以观察到太阳周围的明亮光晕圈，如图 1 所示。如图 2 所示，这种光学现象是由太阳光线在卷层云中的冰晶折射而产生的，该云层高度约 5.5 km。

To understand the phenomenon of the halo, we simplify the problem in two dimensions. In the following, we denote the angle of incidence on an ice crystal θ_i, the angle of refraction at the first interface θ_2, the angle of refraction at the exit of the crystal θ_o , and the angle of deflection between the ingoing and the outgoing sun ray θ_D.

为了理解光晕现象，我们将问题简化为两维。在下文中，我们以θ_i表示冰晶上的入射角，θ_2表示为经过第一个界面的折射角，θ_0表示为光线离开晶体的折射角，以及θ_D表示为入射和出射太阳光线之间的偏转角。

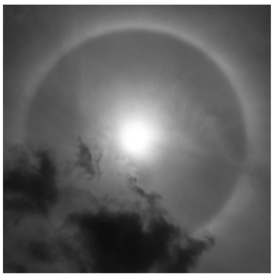


Fig. 1: A halo surrounding the sun
图1：太阳周围的光晕

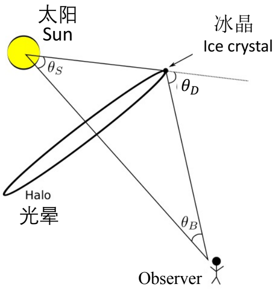


Fig. 2: Formation of a halo.
图2：晕圈的形成。

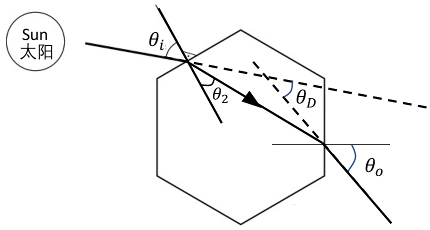


Fig. 3: Light refracted by an ice crystal
图3：由冰晶折射的光

(a) We consider an ice crystal in the form of a regular hexagon (Fig. 3). Derive an expression for θ_D as a function of θ_2, n_{Air} and n_{ice}, where n_i denotes the refractive index of the medium i. ($n_{Air} = 1$, $n_{ice} \approx 1.31$) [3]
(a) 我们考虑一个正六边形的冰晶(图 3)。试求出θ_D的表达式，并以θ_2, n_{Air}和n_{ice}作为其函数表示，其中n_i表示介质i的折射率。($n_{Air} = 1$，$n_{ice} \approx 1.31$) [3]

(b) Estimate the angular radius of the halo as measured by the observer on the ground. [2]
The identity may be useful:
试估算观察者在地面上看到光晕的角半径大小。[2]
这个公式可能有用:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

(c) A closer look at a halo reveals the light spectrum along its entire circumference. Which of the colors, red or blue, is on the inner, which on the outer side of the halo? [1]
仔细观察一个光晕，可以在圆周上观察到不同颜色。问光晕的内侧是哪一种颜色，红色或蓝色，在光晕的外侧又是哪一种颜色？[1]

Solution:
By Snell's law,

$$n_{air}\sin\theta_i = n_{ice}\sin\theta_2 \rightarrow \sin\theta_i = n_{ice}\sin\theta_2$$
$$n_{ice}\sin\theta_3 = \sin\theta_o$$
$$\theta_2 + \theta_3 = \frac{\pi}{3}$$
$$\theta_D = (\theta_i - \theta_2) + (\theta_o - \theta_3) = \theta_i - \frac{\pi}{3} + \theta_0$$
$$\theta_r = \arcsin\left(n_{ice}\sin\left(\frac{\pi}{3} - \arcsin\left(\frac{1}{n_{ice}}\sin\theta_i\right)\right)\right)$$
$$\theta_D = \arcsin(n_{ice}\sin\theta_2) - \frac{\pi}{3} + \arcsin\left(n_{ice}\sin\left(\frac{\pi}{3} - \theta_2\right)\right)$$

(b) The halo is observed at the stationary point of θ_D

$$\frac{d\theta_D}{d\theta_2} = \frac{1}{\sqrt{1-n_{ice}^2\sin^2\theta_2}} n_{ice}\cos\theta_2 - \frac{1}{\sqrt{1-n_{ice}^2\sin^2\left(\frac{\pi}{3}-\theta_2\right)}} n_{ice}\cos\left(\frac{\pi}{3}-\theta_2\right) = 0$$

$$\rightarrow \theta_2 = \frac{\pi}{3} - \theta_2 \quad i.e.\ \theta_2 = \frac{\pi}{6}$$

and the corresponding θ_D is:

$$\theta_{D,\min} = 21.8°$$

The angular radius of the halo is $\theta_B = \theta_{D,\min} - \theta_S \approx \theta_{D,\min} = 21.8°$

(c) The refractive index of red color is smaller than that of blue. The deflection angle corresponding to the blue color should be bigger and hence the outer of the halo should be blue, and inner of the halo is red.