Problem 1: Oscillations of the Sun (22 points) 太阳的振荡（22 分）

The sun is made of compressible gas. It can oscillate in a variety of ways. Investigating these oscillations has provided rich information on the interior of the Sun. In this problem we study two kinds of waves: pressure waves and gravity waves.
太阳的成份是可压缩气体。它可以以各种方式振荡。研究这些振荡提供了有关太阳内部的丰富信息。在这问题中，我们研究两种波：压力波和重力波。

Part A. Pressure Waves (15 points) 压力波（15 分）

Most of us are familiar with sound waves propagating through Earth's atmosphere, which is a pressure wave. In the Sun, however, we need to consider the fact that gas density falls off with height because of gravity. In this problem, we will use the following notations:
我们大多数人都熟悉在地球大气层传播的声波，它是一种压力波。但是在太阳内，我们需要考虑由于重力而导致气体密度随高度下降。在这个问题中，我们将采用以下符号：

$\bar{m}$ = average mass of particles 粒子平均质量
g = gravitational acceleration 重力加速度
k_B= Boltzmann constant 波尔兹曼常数
T = absolute temperature 绝对温度
γ = ratio of the constant-pressure specific heat to the constant-volume specific heat 定压比热与定容比热之比

We model the Sun as an atmosphere whose density falls off with height because of gravity. For a thin layer of the atmosphere between heights x and $x + dx$, the equilibrium pressure at these locations are $P(x)$ and $P(x + dx)$ respectively. Assume that the gravitational acceleration and the temperature are constant.
我们将太阳模拟为一个大气层，其密度因重力而随高度下降。对于高度在x 和$x + dx$之间的薄层气体来说，这些位置的稳态压力分别为$P(x)$和$P(x + dx)$。假定重力加速度和温度是恒定的。

A1	Derive the differential equation for the atmospheric density $\rho(x)$. 试推导大气密度$\rho(x)$的微分方程。	**2 points** **2 分**

Let A be the cross-section area of a column of the atmosphere.
External force acting on the gas = $[P(x) - P(x + dx)]A$.
Weight of the gas = $[\rho(x)Adx]g$.
Condition for equilibrium: $[P(x) - P(x + dx)]A = [\rho(x)Adx]g$.
In the limit $dx \to 0$, $P(x + dx) = P(x) + \frac{dP}{dx}dx$.
Hence $\frac{dP}{dx} = -\rho(x)g$.
Gas law: $P = \frac{\rho k_B T}{\bar{m}}$.
$\frac{d\rho}{dx} = -\frac{\bar{m}g}{k_B T}\rho$.

A2	The scale height H of the atmosphere is the height through which the density	**1 points**

	becomes a factor of e^{-1} of the original density. Derive the expression of H. 大气的标度高度H是密度为起始密度e^{-1}倍的高度。求H的表达式。	**1 分**

The solution of the differential equation is $\rho(x) = \rho(0)\exp\left(-\frac{\bar{m}g}{k_B T}x\right)$.
Hence $\frac{\bar{m}g}{k_B T}H = 1 \Rightarrow H = \frac{k_B T}{\bar{m}g}$.

When a pressure wave propagates vertically in the atmosphere, the particles will experience small vertical displacements. Let $u(x,t)$ denote the vertical displacement of the gas particles at time t whose undisturbed position is x.
当压力波在大气中垂直传播时，粒子将经历细小的垂直位移。设$u(x,t)$为气体粒子在時間t時的垂直位移，x为其不受干扰时的位置。

A3	As shown in the Fig. 1, there is a change in thickness of the thin layer. Express the change in thickness in terms containing the gradient $\partial u/\partial x$. (Remark: For u being a function of both x and t, $\partial u/\partial x$ is called the partial derivative of u with respect to x with t taken to be constant.) 如图 1 所示，薄层的厚度有变化。试以梯度$\partial u/\partial x$表示厚度变化。 （备注：u作为x和t二者的函数，$\partial u/\partial x$被称为u相对于x的偏导数，其中t在求导过程中视为常数。）	**1 points** **1 分**

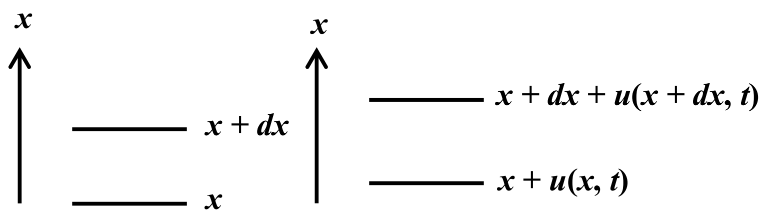


Fig. 1: The vertical displacements of a thin layer of gas particles caused by the propagation of a pressure wave. Note the change in the thickness of the layer.
图 1：由压力波传播引起薄层气体粒子的垂直位移。 请注意层厚度的变化。

Change in thickness $= u(x+dx,t) - u(x,t) = \frac{\partial u}{\partial x}dx$.

A4	In turn, the vertical displacements produce small fluctuations in density and pressure, denoted as $\delta\rho(x,t)$ and $\delta P(x,t)$ respectively. Express the change in $\delta\rho(x,t)$ and $\delta P(x,t)$ in terms containing the gradient $\partial u/\partial x$. Assume that the heat transfer is negligible during the period of the pressure wave. 随之而来，垂直位移产生密度和压力的細小波动，分别表示为$\delta\rho(x,t)$和$\delta P(x,t)$ 。求$\delta\rho(x,t)$和$\delta P(x,t)$的表示式（以梯度$\partial u/\partial x$表示）。假设在压力波传播期间传热可以忽略不计。	**2 points** **2 分**

Since the density is inversely proportional to the thickness of the layer,
$\frac{\delta\rho(x,t)}{\rho(x)} = -\frac{\partial u}{\partial x} \Rightarrow \delta\rho(x,t) = -\rho(x)\frac{\partial u}{\partial x}$.
Since for adiabatic processes, $P \propto \rho^\gamma$,
$\frac{\delta P(x,t)}{P(x)} = \gamma\frac{\delta\rho(x,t)}{\rho(x)} = -\gamma\frac{\partial u}{\partial x} \Rightarrow \delta P(x,t) = -\gamma P(x)\frac{\partial u}{\partial x}$.

A5	Derive the differential equation of motion for $u(x,t)$. Simplify your expressions using the speed of sound $c_s = \sqrt{\gamma P/\rho}$. 试推导$u(x,t)$的微分运动方程。以音速$c_s = \sqrt{\gamma P/\rho}$简化你的答案。	**3 points** **3 分**

Let A be the cross-section area of a column of the atmosphere.

$\rho(x)Adx\frac{\partial^2 u}{\partial t^2} = [\delta P(x,t) - \delta P(x+dx,t)]A.$

$\rho(x)\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x}\delta P(x,t) = \frac{\partial}{\partial x}\left[\gamma P(x)\frac{\partial u}{\partial x}\right].$

$\rho(x)\frac{\partial^2 u}{\partial t^2} = \gamma\frac{\partial P(x)}{\partial x}\frac{\partial u}{\partial x} + \gamma P(x)\frac{\partial^2 u}{\partial x^2}.$

From A1, $\frac{dP}{dx} = -\rho(x)g$. Hence

$\frac{\partial^2 u}{\partial t^2} = -\gamma g\frac{\partial u}{\partial x} + c_s^2\frac{\partial^2 u}{\partial x^2}.$

A6	Show that the solution of the equation of motion is equivalent to a pressure wave traveling through a uniform medium when the wavelength is shorter than a length scale. Derive this length scale. 试证明当波长短于某长度尺度时，运动方程的解等价于穿过均匀介质的压力波。求这个长度尺度。	**2 points** **2 分**

The solution of the differential equation is equivalent to a pressure wave traveling through a uniform medium if the first term is negligible compared with the second term.

First term $= \gamma g\frac{\partial u}{\partial x} \sim \frac{\gamma g}{\lambda}$

Second term $= c_s^2\frac{\partial^2 u}{\partial x^2} \sim \frac{c_s^2}{\lambda^2}$

$\frac{\gamma g}{\lambda} \ll \frac{c_s^2}{\lambda^2} \Rightarrow \lambda \ll \frac{c_s^2}{\gamma g} = \frac{P}{\rho g} = \frac{k_B T}{\bar{m} g} = H.$

Next we seek a sinusoidal wave with angular frequency ω. The energy density of the wave, $\frac{1}{2}\rho\omega^2 u^2$, is expected to remain constant as the wave propagates upward with constant velocity in the direction of decreasing density $\rho(x)$. With this expectation in mind we let
接下来我们寻找一个角频率ω的正弦波。当波沿密度$\rho(x)$减小的方向以恒定速度向上传播时，我们预期波的能量密度$\frac{1}{2}\rho\omega^2 u^2$保持不变。因此我们可设

$$u(x,t) = \frac{f(x)}{\sqrt{\rho(x)}}e^{-i\omega t}.$$

A7	Derive the differential equation for $f(x)$. 试推导$f(x)$的微分方程。	**3 points** **3 分**

$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u = -\omega^2\frac{f(x)}{\sqrt{\rho(x)}}e^{-i\omega t}.$

$\frac{\partial u}{\partial x} = \frac{f\prime(x)}{\sqrt{\rho(x)}}e^{-i\omega t} - \frac{f(x)\rho'(x)}{2\rho(x)^{\frac{3}{2}}}e^{-i\omega t}.$

From A1, $\rho'(x) = -\frac{\rho(x)}{H}$. Hence $\frac{\partial u}{\partial x} = \frac{f\prime(x)}{\sqrt{\rho(x)}}e^{-i\omega t} + \frac{f(x)}{2H\sqrt{\rho(x)}}e^{-i\omega t}.$

$\frac{\partial^2 u}{\partial x^2} = \frac{f''(x)}{\sqrt{\rho(x)}}e^{-i\omega t} + \frac{f'(x)}{H\sqrt{\rho(x)}}e^{-i\omega t} + \frac{f(x)}{4H^2\sqrt{\rho(x)}}e^{-i\omega t}.$

Substituting into A6,

$$-\omega^2 f(x) = -\gamma g f'(x) - \frac{\gamma g}{2H} f(x) + c_s^2 \left[f''(x) + \frac{f'(x)}{H} + \frac{f(x)}{4H^2} \right].$$

$$f''(x) + \left(\frac{1}{H} - \frac{\gamma g}{c_s^2}\right) f'(x) + \left(\frac{1}{4H^2} - \frac{\gamma g}{2Hc_s^2} + \frac{\omega^2}{c_s^2}\right) f(x) = 0.$$

$$f''(x) + \left(\frac{\omega^2}{c_s^2} - \frac{1}{4H^2}\right) f(x) = 0.$$

A8	When the frequency of the pressure wave is below a critical frequency ω_c below the Sun's surface, it becomes trapped inside the Sun. What is ω_c? 当太阳表面下的压力波频率低于临界频率ω_c时，它会被困于太阳内部。求 ω_c。	**1 points** **1 分**

The pressure wave cannot propagate if $\frac{\omega^2}{c_s^2} - \frac{1}{4H^2} < 0 \Rightarrow \omega < \frac{c_s}{2H}$ Hence $\omega_c = \frac{c_s}{2H}$.

Part B. Gravity Waves (7 points) 重力波（7 分）

In Part A we only included the restoring force due to the fluctuation in the pressure gradient for pressure waves traveling in the vertical direction of the Sun's atmosphere. However, for gravity waves propagating in a horizontal direction of the Sun's atmosphere, the buoyancy of the gas may also give rise to a restoring force which can sustain oscillations.
在 A 部，我们只考虑了由于压力波在太阳大气垂直方向传播的压力梯度的波动而产生的恢复力。然而，对于沿太阳大气的水平方向传播的重力波，气体的浮力也可成为维持振荡的恢复力。

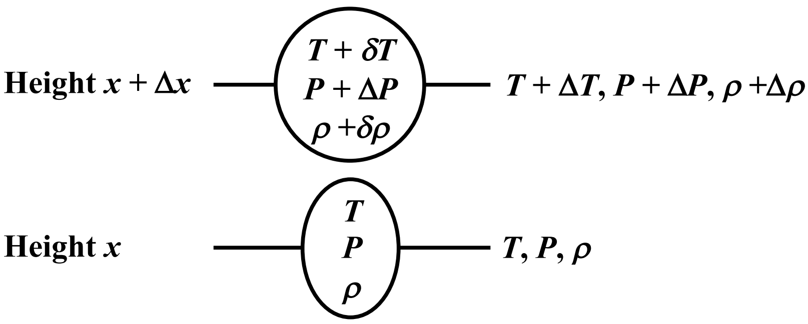


Fig. 2: Displacement of a pocket of gas from height x to height $x + \Delta x$.
图 2：一小口气体从高度x到高度$x + \Delta x$的位移。

To understand this, we consider a small vertical displacement of a pocket of gas in an environment of the same gas with both gradients in the temperature and pressure. As shown in Fig. 2, this pocket of gas has the same temperature, pressure and density as the surrounding gas. When its height is displaced by Δx, it enters an environment with temperature, pressure and density given by $T + \Delta T$, $P + \Delta P$ and $\rho + \Delta\rho$ respectively.
为了理解这一点，我们考虑一小口气体在相同气体的环境中，在温度梯度和压力梯度的影响下作垂直位移。如图 2 所示，这小口气体与周围气体具有相同的温度、压力和密度。当高度移动Δx时，它将进入温度为$T + \Delta T$、压力为$P + \Delta P$和密度为$\rho + \Delta\rho$ 的环境。

For the pocket of gas, the pressure inside the pocket responds rapidly to the environment so that its pressure also changes by ΔP. On the other hand, the change in temperature and

density may be different. Suppose the temperature, pressure and density of the pocket of gas in the new environment are $T + \delta T$, $P + \Delta P$ and $\rho + \delta\rho$ respectively. Assume that there is insufficient time for heat conduction from the pocket of gas to the environment.
对这小口气体而言，内部压力迅速回应环境，使其压力也随之改变为$P + \Delta P$。另一方面，温度和密度的变化可能不同。假设新环境中这小口气体的温度为$T + \delta T$、压力为$P + \Delta P$和密度为$\rho + \delta\rho$。假设没有足够的时间从小口气体向环境传导热量。

B1	Express $\Delta\rho$ and $\delta\rho$ in terms of expressions containing ΔT and ΔP. 求$\Delta\rho$ 和$\delta\rho$的表达式(用ΔT和ΔP表示)。	**2 points** **2 分**

Since the surrounding gas satisfies the ideal gas law $P = \frac{\rho k_B T}{\bar{m}}$, we have $\rho \propto P/T$ and
$\frac{\Delta\rho}{\rho} = \frac{\Delta P}{P} - \frac{\Delta T}{T} \Rightarrow \Delta\rho = \rho\left(\frac{\Delta P}{P} - \frac{\Delta T}{T}\right)$.
Since the pocket of gas undergoes an adiabatic process, $P \propto \rho^\gamma$,
$\frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{\Delta P}{P} \Rightarrow \delta\rho = \frac{\rho}{\gamma}\frac{\Delta P}{P}$.

B2	Suppose the temperature and pressure gradients of the surrounding gas are dT/dx and dP/dx respectively. Derive the equation of motion of the pocket of gas. 假设周围气体的温度梯度和压力梯度分别为dT/dx和dP/dx。试推导这小口气体的运动方程。	**2 points** **2 分**

Let m be the mass of the pocket of gas.
Buoyancy of the pocket of gas $= \frac{m}{\rho+\delta\rho}(\rho + \Delta\rho)g - mg \approx \frac{mg}{\rho}(\Delta\rho - \delta\rho)$
$= mg\left(\frac{\gamma-1}{\gamma}\frac{\Delta P}{P} - \frac{\Delta T}{T}\right)$.
Applying Newton's law of motion,
$m\frac{d^2}{dt^2}\Delta x = mg\left(\frac{\gamma-1}{\gamma}\frac{\Delta P}{P} - \frac{\Delta T}{T}\right)$
$\frac{d^2}{dt^2}\Delta x = g\left(\frac{\gamma-1}{\gamma P}\frac{dP}{dx} - \frac{1}{T}\frac{dT}{dx}\right)\Delta x$

B3	Determine the range of temperature gradient dT/dx in which the pocket of gas can exhibit oscillations. Express the bound(s) of the temperature gradient in terms of T/H. 求这小口气体可出现振荡的温度梯度dT/dx范围(以T/H表示这范围的界限)。	**2 points** **2 分**

Oscillations are possible when $\frac{\gamma-1}{\gamma P}\frac{dP}{dx} - \frac{1}{T}\frac{dT}{dx} < 0$.
$-\frac{dT}{dx} < -\frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dx}$.
From A1, $\frac{dP}{dx} = -\rho g \Rightarrow -\frac{dT}{dx} < \frac{\gamma-1}{\gamma}\frac{\rho g T}{P} = \frac{\gamma-1}{\gamma}\frac{\bar{m}g}{k_B} = \frac{\gamma-1}{\gamma}\frac{T}{H}$.

B4	How does the gas in the Sun behave when the temperature gradient is outside the range considered in B3? 当温度梯度超出 B3 考虑的范围时，太阳中的气体会有什么行为？	**1 points** **1 分**

The gas will become **convective**.

Reference for this problem: A. C. Phillips, The Physics of Stars, 2nd edition (Wiley, 1994)

END of Problem 1
问题 1 完

Problem 2: Plasmon Resonance and SERS 等离子共振和 SERS

Surface-enhanced Raman spectroscopy (SERS) is one of the most prominent optical phenomena in the last 40 years. SERS is based on **plasmon resonance**, referring to the significant increase in electric field intensity near the small metal granules under certain conditions. In order to determine these conditions, it is necessary to learn how to describe the properties of metals placed in oscillating electromagnetic fields.
表面增强拉曼光谱（SERS）是近 40 年来最重要的光学现象之一。 SERS 的基础是**等离子共振**，指在某些条件下小金属颗粒附近的电场强度显着增加。为了确定这些条件，我们有必要认识如何描述放置在振荡电磁场中的金属特性。

Properties of a medium in an electric field are described as follows:

$$\vec{D} = \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

where $\vec{E}$ and $\vec{D}$ are the electric field intensity and the electric displacement respectively, ϵ is the permittivity of the medium, $\vec{P}$ is the electric polarization (electric dipole moments per unit volume), ϵ_0 is the vacuum permittivity. The boundary conditions in the absence of free charges are the continuity of electric field tangential to the boundary and the continuity of the electric displacement normal to the boundary.
介质在电场中的特性描述如下：

$$\vec{D} = \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

其中$\vec{E}$ 和$\vec{D}$分别是电场强度和电位移，ϵ是介质的介电常数，$\vec{P}$是电极化强度（单位体积的电偶极矩），ϵ_0是真空介电常数。在没有自由电荷下的边界条件分别是与边界相切的电场的连续性和垂直于边界的电位移的连续性。

In an oscillating electromagnetic field, the permittivity of a media (including metals) is dependent on the electromagnetic field frequency, $\epsilon = \epsilon(\omega)$.
在振荡电磁场中，介质（包括金属）的介电常数取决于电磁场频率：$\epsilon = \epsilon(\omega)$。

Part A. Free Electron Gas (3 points) 自由电子气体（3 分）

Consider a metal occupying an infinite space. Positive ions form a crystal lattice. Free electrons move inside the lattice. The number density of positive ions and electrons are the same and equal to n.
考虑一占据无限空间的金属。正离子形成晶格。自由电子在晶格内移动。正离子和电子的数密度相同并等于n 。

A uniform oscillating electric field $\vec{E}_0\sin(\omega t)$ is applied in the metal. Assume that the ions are infinitely heavy and fixed. The effective mass and charge of an electron are denoted as m and $-e$ respectively. Within the simple framework of the free electron approximation one can assume that the field acting on an electron is equivalent to $\vec{E}_0\sin(\omega t)$. All other forces (including dissipative forces) are small and negligible.
在金属中施加均匀的振荡电场$\vec{E}_0\sin(\omega t)$。假设离子无限重并且固定。电子的有效质量和电荷分别表示为m和$-e$。 在自由电子近似的简单框架内，可以假定作用于电子的场等价于$\vec{E}_0\sin(\omega t)$。 所有其他力（包括耗散力）都很小并且可以忽略不计。

A1	The electric field drives the collective motion of the electrons $\vec{r}(t)$ along the electric field direction. Derive the expressions of $\vec{r}(t)$ and the polarization $\vec{P}(t)$ at the steady state. 电场驱动电子沿电场方向作集体运动$\vec{r}(t)$。试推导在稳定状态下的$\vec{r}(t)$和电极化强度$\vec{P}(t)$的表达式。	**2 points** **2 分**

Newton's second law of motion for electron,

$m\frac{d^2}{dt^2}\vec{r}(t) = -e\vec{E}_0\sin(\omega t)$

One can get

$\vec{r}(t) = \frac{e\vec{E}_0}{m\omega^2}\sin(\omega t)$

Since the heavy (position) ion does not move, the electric dipole moment per atom is

$\vec{p} = -e\vec{r}(t) \rightarrow \vec{P}(t) = n\vec{p} = -\frac{ne^2\vec{E}_0}{m\omega^2}\sin(\omega t)$

A2	Determine the metal permittivity $\epsilon(\omega)$. 求金属的介电常数$\epsilon(\omega)$。	**1 points** **1 分**

$\epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P} \rightarrow \epsilon\epsilon_0\vec{E} = \epsilon_0\vec{E} - \frac{ne^2}{m\omega^2}\vec{E} \rightarrow \epsilon = 1 - \frac{ne^2}{\epsilon_0 m\omega^2}$

Part B. Plasmon Resonance (16 points) 等离子共振（16 分）

In this part we consider a dielectric sphere of radius R and permittivity ϵ in a uniform electric field $\vec{E}_0$. Due to the polarization of the dielectric material, the electric field in the sphere and its neighborhood is modified. The polarization of the dielectric sphere is due to mobile charges being shifted in the uniform electric field. Here we model the dielectric effects by two oppositely charged spheres with radius R and charge density $\pm\rho$ being displaced along $\vec{E}_0$ by displacements $\pm\delta/2$ respectively (see Fig. 1).

在这部分我们考虑在均匀电场$\vec{E}_0$中半径为R和介电常数为ϵ的介电球。由于介质材料的极化作用，球体及其附近的电场被改变了。介质球的极化是由于电荷在均匀电场中产生移位。在这里，我们通过两个带相反电荷的球体来模拟介电效应。两个电荷球的半径为R、电荷密度分别为$\pm\rho$，沿$\vec{E}_0$方向的移位分别为$\pm\delta/2$（见图 1）。

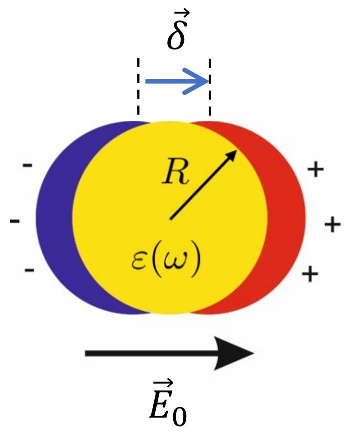


Fig. 1: Dielectric spheres in the uniform electric field.
图 1：在均匀电场中的介电球。

B1	The total electric field $\vec{E}_{in}$ inside the dielectric sphere is the superposition of the external field $\vec{E}_0$ and the electric fields due to the two charged spheres. Derive an expression for $\vec{E}_{in}$ in terms of $\vec{E}_0$ and the polarization $\vec{P}$ due to the two charged spheres. 介电球内部的总电场$\vec{E}_{in}$是由外加电场$\vec{E}_0$和两个带电球体引起的电场的叠加。试推导$\vec{E}_{in}$的表达式(以$\vec{E}_0$和两个带电球体引起的电极化强度$\vec{P}$表示)。	**2 points** **2 分**

Electric field at position $\vec{r}$ due to the positively charged sphere:

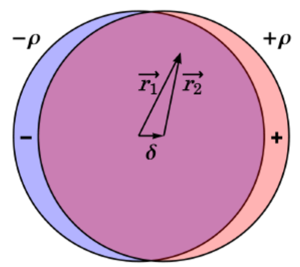
$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{4\pi r_2^3}{3}\rho\right)\frac{\vec{r}_2}{r_2^3} = \frac{\rho}{3\epsilon_0}\vec{r}_2$ where $\vec{r}_2 = \vec{r} - \frac{\vec{\delta}}{2}$

Similarly, electric field due to the negatively charged sphere:

$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{4\pi r_1^3}{3}\rho\right)\frac{\vec{r}_1}{r_1^3} = \frac{\rho}{3\epsilon_0}\vec{r}_1$ where $\vec{r}_1 = \vec{r} + \frac{\vec{\delta}}{2}$

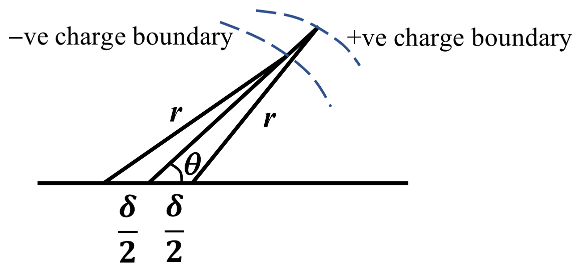
Hence

$$\vec{E}_{in} = \vec{E}_0 + \vec{E}_2 - \vec{E}_1 = \vec{E}_0 + \frac{\rho}{3\epsilon_0}(\vec{r}_2 - \vec{r}_1) = \vec{E}_0 - \frac{\rho\vec{\delta}}{3\epsilon_0} = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0}.$$



B2	Due to the dielectric effects, surface charge is induced on the surface of the dielectric sphere. Consider a point on the surface of the dielectric sphere where the outward unit vector normal to the spherical surface is denoted as $\hat{n}$. In the limit of $\delta << R$, derive an expression for the induced surface charge density σ at this point in terms of the polarization $\vec{P}$ and $\hat{n}$. 由于介电效应，介电球表面上产生感应的表面电荷。考虑电介球表面上的一个点，它与球形表面垂直的向外单位矢量表示为$\hat{n}$。在$\delta << R$的极限下，试推导表面感应电荷密度的表达式(以电极化强度$\vec{P}$和$\hat{n}$表示)。	**2 points** **2 分**

Consider a point making an angle θ at the center of the sphere with the electric field. As shown in the figure, distance of the boundary of the positively charged sphere from the center $\approx r + \frac{\delta}{2}\cos\theta$.



Similarly, distance of the boundary of the negatively charged sphere from the center $\approx r - \frac{\delta}{2}\cos\theta$.

Hence, thickness of the layer of the induced charge

$\approx \left(r + \frac{\delta}{2}\cos\theta\right) - \left(r - \frac{\delta}{2}\cos\theta\right) = \delta\cos\theta.$

Surface charge density: $\sigma \approx \rho\delta\cos\theta = P\cos\theta = \vec{P}\cdot\hat{n}.$

B3	Following B2, derive the relation between the normal components of the electric fields $\vec{E}_{out}\cdot\hat{n}$ and $\vec{E}_{in}\cdot\hat{n}$ at the surface of the dielectric sphere.	**1 point** **1 分**

	根据 B2，试推导电场的法向分量$\vec{E}_{out}\cdot\hat{n}$和$\vec{E}_{in}\cdot\hat{n}$在介电球表面间的关系。	

Using Gauss' law, $\vec{E}_{out}\cdot\hat{n} = \vec{E}_{in}\cdot\hat{n} + \frac{\sigma}{\epsilon_0} = \vec{E}_{in}\cdot\hat{n} + \frac{\vec{P}\cdot\hat{n}}{\epsilon_0}$.

B3	Express the induced electric dipole moment $\vec{d}_0$ of the dielectric sphere as a function of $\vec{E}_0$. 求介电球的感应电偶极矩$\vec{d}_0$(作为$\vec{E}_0$的函数表示)。	**3 points** **3 分**

From B.1, $\vec{E}_{in}\cdot\hat{n} = \vec{E}_0\cdot\hat{n} - \frac{\vec{P}\cdot\hat{n}}{3\epsilon_0}$.

From B.3, $\vec{E}_{out}\cdot\hat{n} = \vec{E}_{in}\cdot\hat{n} + \frac{\vec{P}\cdot\hat{n}}{\epsilon_0}$.

Eliminating the polarization, $\vec{E}_{out}\cdot\hat{n} = 3\vec{E}_0\cdot\hat{n} - 2\vec{E}_{in}\cdot\hat{n}$

Consider the boundary condition: $\vec{E}_{out}\cdot\hat{n} = \epsilon\vec{E}_{in}\cdot\hat{n}$.

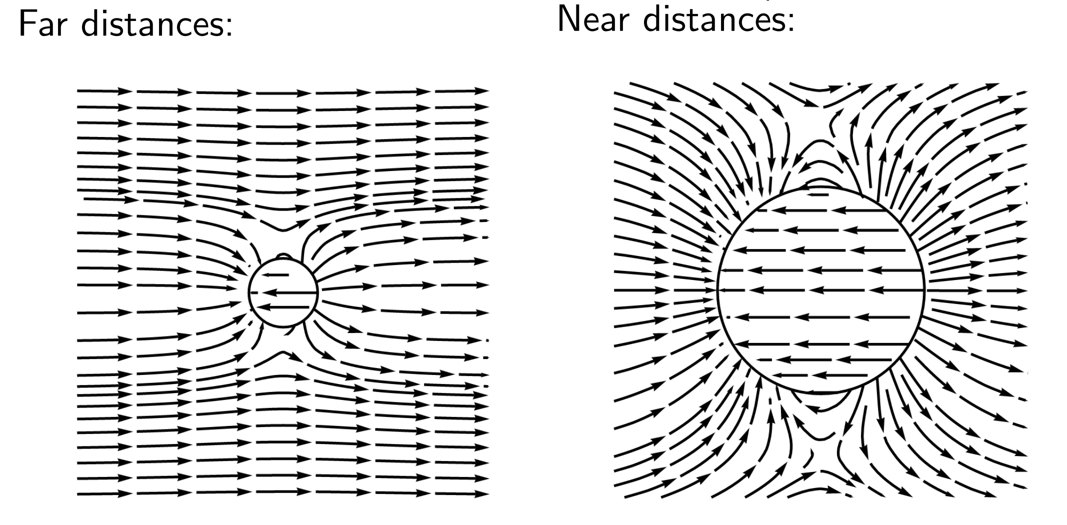
Eliminating $\vec{E}_{out}$: $\vec{E}_{in}\cdot\hat{n} = \frac{3}{\epsilon+2}\vec{E}_0\cdot\hat{n}$ and $\vec{P}\cdot\hat{n} = 3\epsilon_0(\vec{E}_0\cdot\hat{n} - \vec{E}_{in}\cdot\hat{n}) = 3\epsilon_0\left(\frac{\epsilon-1}{\epsilon+2}\right)\vec{E}_0\cdot\hat{n}$

Since $\vec{P}$ and $\vec{E}_0$ are parallel, $\vec{P} = 3\epsilon_0\left(\frac{\epsilon-1}{\epsilon+2}\right)\vec{E}_0$ and $\vec{d}_0 = \frac{4}{3}\pi R^3\vec{P} = 4\pi R^3\epsilon_0\left(\frac{\epsilon-1}{\epsilon+2}\right)\vec{E}_0$.

Let us analyze the behavior of a metal sphere in an oscillating electric field of angular frequency ω and amplitude $\vec{E}_0$. The radius of the sphere is R. When the wavelength and field penetration depth are both much greater than the size of the sphere, one can consider the metal sphere as a dielectric in a uniform electric field, except that one has to use $\epsilon(\omega)$ (analogous to the one expressed in the previous part) in place of the permittivity. Hence the external electric field is $\vec{E} = \vec{E}_0\cos\omega t$, and the dipole moment is $\vec{d} = \vec{d}_0\cos\omega t$.

让我们分析在角频率为ω、振幅为$\vec{E}_0$的振荡电场下金属球的行为。 球体的半径是R。当波长和电场穿透深度都远大于球体的尺度时，可以将金属球看作均匀电场中的电介质，除了必须使用$\epsilon(\omega)$（类似于前面部分中所表达的那种） 代替介电常数。 因此，外部电场是$\vec{E} = \vec{E}_0\cos\omega t$，偶极矩是$\vec{d} = \vec{d}_0\cos\omega t$ 。

B5	Sketch qualitatively the field lines (inside, near and far from the ball) in the system assuming $\epsilon(\omega) = -3$. 假设$\epsilon(\omega) = -3$，定性描绘系统中的场线（球内、附近和远距）。	**2 points** **2 分**



For $\epsilon(\omega) = -3$, $\vec{E}_{in} \cdot \hat{n} = \frac{3}{\epsilon+2}\vec{E}_0 \cdot \hat{n} = -3\vec{E}_0 \cdot \hat{n}$, and $\vec{E}_{out} \cdot \hat{n} = 9\vec{E}_0 \cdot \hat{n}$.
The sketch should show the following features:
In distant regions, the field lines are parallel to $\vec{E}_0$.
In the upper and lower regions the field lines are continuous from left to right.
In the neighborhood of the sphere the field lines terminate or originate at the sphere.
The direction at which the field lines terminate or originate at the sphere lies on the equatorial side of the normal.
Near the equatorial plane the field lines originate and terminate at the sphere in the direction opposite to $\vec{E}_0$.
There is a point of zero field above and below the sphere.
Inside the dielectric sphere the field lines are antiparallel to $\vec{E}_0$.

B6	When $\omega = \omega_{res}$, resonance takes place and the internal electric intensity $\lvert\vec{E}_{in}\rvert$ increases to infinity. Determine ϵ_{res}, the value of $\epsilon(\omega)$ when $\omega = \omega_{res}$. 当$\omega = \omega_{res}$时，共振发生並且内部电场强度增加到无穷大。求ϵ_{res}，即$\epsilon(\omega)$在 $\omega = \omega_{res}$时的值。	**1 points** **1 分**

$\vec{E}_{in} = \frac{3}{\epsilon+2}\vec{E}_0 \rightarrow \epsilon = -2$

Significant increase in the electric field amplitude with frequency equaling ω_{res} is called the **plasmon resonance**. Assuming that there is no power dissipation, $\lvert\vec{E}_{in}\rvert$ approaches infinity. Taking into account dissipation, the major loss of power comes from dipole radiation.
电场振幅随着频率接近ω_{res}而显着增加, 这现象称为**等离子共振**。假设没有功耗，$\lvert\vec{E}_{in}\rvert$趋向无穷大。考虑到耗散，功率的主要损失来自偶极辐射。

B7	An oscillating dipole emits energy. Estimate the power I of this energy loss using dimensional analysis. A dipole radiation intensity depends on the dipole moment amplitude $\lvert\vec{d}_0\rvert$, its oscillation frequency ω_{res}, speed of light c and vacuum permittivity ϵ_0. 振荡偶极子发射能量。试用*量纲*分析来估算这种能量损失的功率 I 。偶极子辐射强度取决于偶极矩振幅$\lvert\vec{d}_0\rvert$、振荡频率ω_{res}、光速c和真空介电常数ϵ_0。	**3 points** **3 分**

Using the dimensional analysis we obtain:

$I = \epsilon_0^{\alpha} \left|\vec{d}_0\right|^{\beta} \omega_{res}^{\gamma} c^{\eta}$

$$[I] = N \cdot m \cdot s^{-1}$$

$$[\epsilon_0] = C^2 \cdot N^{-1} \cdot m^{-2},\ \left[\left|\vec{d}_0\right|\right] = C \cdot m,\ [\omega_{res}] = s^{-1}, \qquad [c] = m \cdot s^{-1}$$

Solving the equations, we get

$I = \frac{\left|\vec{d}_0\right|^2 \omega_{res}^4}{\epsilon_0 c^3}$ [2.0 : 0.5 for each exponents]

B8	In practice, $\left|\vec{E}_{in}\right|$ is finite due to power dissipation at the plasmon resonance frequency ω_{res}. Suggest an approximate expression of the internal electric field intensity $\left|\vec{E}_{in}\right|$ using the condition that the power output is balanced by the mean power pumped into the system by the external field during plasmon resonance. 实际上，由于等离子共振频率下的功率损耗，$\left|\vec{E}_{in}\right|$是有限的。假設等离子共振时功率的输出与外场输入系统的平均功率平衡，試提出内部电场强度$\left|\vec{E}_{in}\right|$的近似表达式。	**2 points** **2 分**

Work done in producing an electric dipole is $E_0\left(\frac{4}{3}\pi R^3 \rho\right)\delta \approx E_0 d_0$.

Power consumed in producing the oscillating electric dipole,

$\frac{d}{dt}\left(\vec{E}_0 \cdot \vec{d}\right) \approx \vec{E}_0 \cdot \frac{d}{dt}\vec{d} \approx E_0 d_0 \omega_{res}$

By energy balance and $\vec{d}_0 = \frac{4\pi}{3} R^3 \epsilon_0 (\epsilon - 1)\vec{E}_{in} \to d_0 \approx \epsilon_0 R^3 E_{in}$, we have

$E_0 d_0 \omega_{res} = \frac{d_0^2 \omega_{res}^4}{\epsilon_0 c^3} \to E_0 = \frac{d_0 \omega_{res}^3}{\epsilon_0 c^3} \to E_{in} = E_0 \left(\frac{c}{\omega_{res} R}\right)^3$

Part C. Raman Spectroscopy (7 points) 拉曼光谱（7 分）

SERS is based on the phenomenon of **Raman scattering**, referring to the interaction of electromagnetic waves with mechanical vibrations of molecules. First we consider a molecule configuration. We assume that a molecule is made up of a number of atoms connected by chemical bonds that behave like springs. Hereafter we consider a diatomic molecule.

SERS 的基础是**拉曼散射现象**，指的是电磁波与分子机械振动的相互作用。 首先我们考虑一个分子的组态。我们假设一个分子是由许多通过化学键连接起来的原子组成的，化学键的作用如同弹簧。我们这里考虑双原子分子。

C1	Consider two masses m_1 and m_2 connected by a spring of spring constant k. Determine the frequency ω_0 of small-amplitude system oscillations. 考虑两个质量m_1和m_2，它们通过弹簧常数为k的弹簧连接。求系统在小振幅时的振荡频率ω_0。	**1 points** **1 分**

Using the reduced mass,

$\omega_0 = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}$

A polyatomic molecule is characterized by its spectrum of resonant frequencies. One can identify the molecule with the knowledge of its spectrum. This is the basic idea of SERS.
多原子分子的特征是其共振频率谱。人们可以通过其频谱的认识来识别分子。这是SERS 的基本思想。

Let us analyze the behavior of a molecule in an external electric field $\vec{E}_0 \cos(\omega t)$. We assume that atoms have no charge, i.e. the molecule has no dipole moment in the absence of the external electric field. However, a molecule is polarized by the external electric field
让我们分析一个分子在外部电场$\vec{E}_0 \cos(\omega t)$中的行为。我们假设原子没有电荷，即在没有外部电场的情况下分子没有偶极矩。然而，分子能被外部电场极化

$$\vec{d} = \epsilon_0 \alpha \vec{E}_0 \cos(\omega t)$$

where α is the polarizability of the molecule. We assume that an induced dipole moment $\vec{d}$ is parallel to the electric field $\vec{E}$. Due to thermal agitation, mechanical oscillations of the molecules always exist at finite temperatures, and we assume that the thermally agitated angular frequency is ω_0.
其中α为分子的极化率。我们假设感应偶极矩$\vec{d}$与电场$\vec{E}$平行。由于热搅动，分子的机械振荡总是存在于有限的温度下，并且我们假定热搅动的角频率是ω_0 。

During molecular oscillations, the distance between atoms in a molecule deviates from its equilibrium value. Suppose the deviation x of the interatomic distance is given by $x = x_0 \cos(\omega_0 t)$. Furthermore, when the interatomic distance changes, the polarizability of the atoms changes accordingly, i.e., $\alpha = \alpha(x)$ (see Fig. 2).
在分子振荡过程中，分子中原子之间的距离偏离其平衡值。假设原子间距离的偏差x由$x = x_0 \cos(\omega_0 t)$ 给出。此外，当原子间距改变时，原子的极化率相应地改变，即$\alpha = \alpha(x)$（见图 2）。

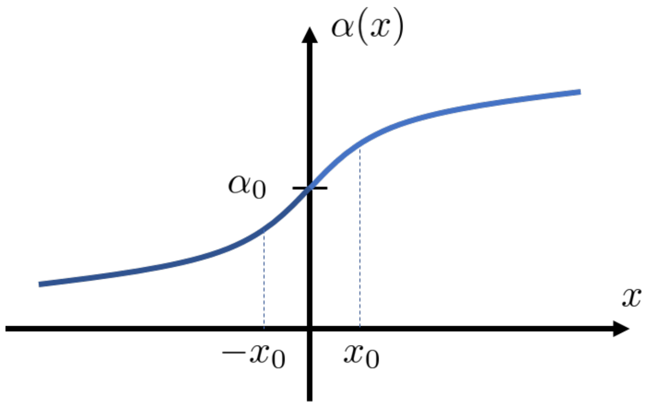


Fig. 2: The dependence of the polarizability of atoms in a molecule on the deviation of the interatomic distance during molecular oscillations.
图 2：分子中原子的极化率对分子振荡过程中原子间距的偏差的依赖性。

C2	Assuming that the amplitude of mechanical oscillations is small, express $\alpha(x)$ using linear approximation, given that $\alpha(0) = \alpha_0$ and $\left.\frac{d\alpha}{dx}\right\|_{x=0} = \beta_0$. 假设机械振荡的幅度很小，试以$\alpha(0) = \alpha_0$ 和 $\left.\frac{d\alpha}{dx}\right\|_{x=0} = \beta_0$ 将 $\alpha(x)$作线	**1 points** **1 分**

	性近似表示。	

$\alpha(x) = \alpha_0 + \beta_0 x$

C3	Determine the diatomic molecule dipole moment in the external field $\vec{E}_0 \cos(\omega t)$. Provide the answer in the form: 求外场$\vec{E}_0 \cos(\omega t)$中的双原子分子偶极矩，答案以下列形式表示： $\vec{d} = \sum_i \vec{d}_i \cos(\omega_i t)$ (1)	**2 points** **2 分**

$\vec{d} = \epsilon_0(\alpha_0 + \beta_0 x_0 \cos(\omega_0 t))\vec{E}_0 \cos(\omega t)$

$\Rightarrow \vec{d} = \epsilon_0 \alpha_0 \vec{E}_0 \cos(\omega t) + \frac{1}{2}\epsilon_0 \beta_0 x_0 \cos(\omega - \omega_0)t + \frac{1}{2}\epsilon_0 \beta_0 x_0 \cos(\omega + \omega_0)t$

C4	A detector logs the dipole radiation of the molecule and detects several peaks with frequencies ω_i, corresponding to expression (1). The height of each peak is equal to the radiation intensity of the dipole $\vec{d}_i$. Determine frequency and height of each peak. Express the answer in terms of $\epsilon_0, \alpha_0, \beta_0, x_0, \omega, \omega_0$ and $\vec{E}_0$. 检测器记录分子的偶极辐射，并检测到数个具有频率ω_i，对应于表达式（1）的峰值。每个峰值的高度等于偶极子$\vec{d}_i$的辐射强度。求每个峰值的频率和高度，以$\epsilon_0, \alpha_0, \beta_0, x_0, \omega, \omega_0$ 和 $\vec{E}_0$表达答案。	**3 points** **3 分**

The dipole radiation intensity reads

$$I = \frac{|\vec{d}_0|^2 \omega_{res}^4}{\epsilon_0 c^3}$$

where ω_{res} is the oscillation frequency. Hence we can observe 3 peaks with amplitudes:

Frequency	Amplitude
ω	$\frac{\epsilon_0(\alpha_0 E_0)^2 \omega^4}{c^3}$
$\omega - \omega_0$	$\frac{\epsilon_0(\beta_0 x_0 E_0)^2 (\omega - \omega_0)^4}{4c^3}$
$\omega + \omega_0$	$\frac{\epsilon_0(\beta_0 x_0 E_0)^2 (\omega + \omega_0)^4}{4c^3}$

[0.5 for each of number in the table]

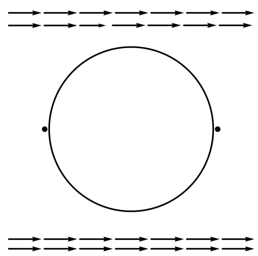
The presence of peaks with frequencies differing from ω in the spectrum is called **Raman scattering**. The stronger the external electric field the higher the signal detected from one molecule. A strong electric field can be obtained using the phenomenon of **plasmon resonance**. This is a difference between SERS and ordinary Raman spectroscopy.

频谱具有不同于ω的峰值的现象，被称为**拉曼散射**。 外部电场越强，从一个分子检测到的信号就越高。 利用**等离子共振**现象可以获得强大的电场。这是 SERS 和普通拉曼光谱之间的一个区别。

Part D. Surface-Enhanced Raman Spectroscopy (SERS) (7 points)
表面增强拉曼光谱（SERS）（7 分）

Consider a sphere of permittivity $\epsilon(\omega)$ in the uniform oscillating electric field of amplitude $|\vec{E}_0|$ in the case of plasmon resonance.
考虑在等离子共振情况下，在均匀振荡电场(振幅为 $|\vec{E}_0|$)中的介电球，其介电常数为$\epsilon(\omega)$。

D1	The molecule of the investigated material has to be placed into the points of maximal electric intensity. Locate these points in the figure on the answer sheet. 被检测材料的分子必须放置在最大电场强度的位置。 请在答题纸的图中标出这些位置。	**2 points** **2 分**



D2	Determine the enhancement factor $g(\omega)$ of the electric field at these points, where $g(\omega) = \max_{\vec{r}} \frac{\lvert\vec{E}(\vec{r})\rvert}{\lvert\vec{E}_0\rvert}$. Express the answer in terms of the metal permittivity $\epsilon(\omega)$. 求在这些位置电场的增强因子$g(\omega)$，其中$g(\omega) = \max_{\vec{r}} \frac{\lvert\vec{E}(\vec{r})\rvert}{\lvert\vec{E}_0\rvert}$。以金属介电常数$\epsilon(\omega)$表示答案。	**1 points** **1 分**

$$g(\omega) = \max_{\vec{r}} \frac{|\vec{E}(\vec{r})|}{|\vec{E}_0|} = \left|\epsilon(\omega)\frac{E_{in}}{E_0}\right| = \left|\frac{3\epsilon(\omega)}{2+\epsilon(\omega)}\right|$$

Metal beads enhance the external electric field radiation of amplitude $\vec{E}_0$ and dipole radiation of the molecule as well. The second process is characterized by the enhancement factor $g'(\omega, \omega_0)$. When $\omega \gg \omega_0$, one can assume $g' \approx g$. Then the signal intensity in SERS is g^4 times greater than that in ordinary Raman spectroscopy.
金属珠增强了外部电场辐射振幅$\vec{E}_0$和分子的偶极辐射。第二个过程以增强因子$g'(\omega, \omega_0)$为特征。 当$\omega \gg \omega_0$时，可假设$g' \approx g$。因此，SERS 中的信号强度比普通拉曼光谱中的信号强度大g^4倍。

Usually the signal comes from many molecules. Dipole radiations of the molecules are not coherent to each other. Thus a total radiation intensity formed by N molecules is equal to NI_0, where I_0 is the intensity of dipole radiation from a single molecule. An example of the experimental data is presented in Fig. 3.
信号通常来自许多分子。分子的偶极辐射彼此不相干。因此，由N个分子形成的总辐射强度为NI_0，其中I_0是来自单个分子的偶极辐射强度。图 3 给出了一组实验数据。

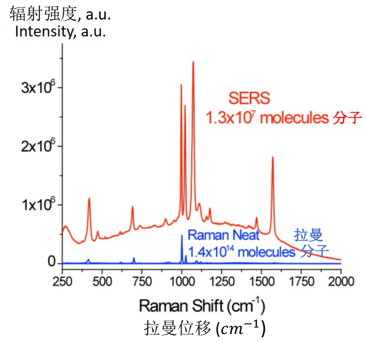


Fig. 3: [Credit: U.S. Naval Research Laboratory] Raman signal spectra. Red curve corresponds to SERS, blue to ordinary Raman spectroscopy. X-axis corresponds to Raman shift $k_0 = \omega_0/c$, and Y-axis corresponds to radiation intensity in arbitrary units. Note the different number of molecules in these experiments.
图 3：[美国海军研究实验室] 拉曼信号光谱。 红色曲线对应于 SERS，蓝色曲线对应于普通拉曼光谱。 X轴对应于拉曼位移（$k_0 = \omega_0/c$），而 Y轴对应于任意单位下的辐射强度。 注意这些实验中分子的数量不同。

D3	By analyzing the experimental data presented in Fig. 3, estimate enhancement factor g due to the plasmon resonance at the peak of Raman shift $\omega_0/c = 1000\ \mathrm{cm}^{-1}$. Assume that $\omega_0 \ll \omega$. 通过分析图 3 中给出的实验数据，试估算在拉曼位移峰值$\omega_0/c = 1000\ \mathrm{cm}^{-1}$处由等离子共振引起的增强因子$g$。假设$\omega_0 \ll \omega$ 。	**2 points** **2 分**

$$\left(\frac{I_{SERS}}{N_{SERS}}\right) / \left(\frac{I_{RAM}}{N_{RAM}}\right) = g^4$$

$$\Longrightarrow g = \sqrt[4]{\left(\frac{I_{SERS}}{N_{SERS}}\right) / \left(\frac{I_{RAM}}{N_{RAM}}\right)} \approx 90$$

[2.0 pt: 80-100, 1.0 pt: 60-120, 0 pt: otherwise]

D4	Based on the results of Part B, estimate the radius R of the metal beads used in the experiment. Assume that the wavelength of the external radiation $\lambda = 785$ nm. 根据 B 部的结果，试估算实验中使用的金属珠的半径R。 假定外部辐射的波长为$\lambda = 785$ nm。	**2 points** **2 分**

$\omega_{res} = 2\pi \frac{c}{\lambda} \rightarrow \frac{c}{\omega_{res}} = \frac{\lambda}{2\pi}$

$\frac{3\epsilon(\omega_{res})}{2+\epsilon(\omega_{res})} \approx 90 \rightarrow \epsilon(\omega_{res}) \approx -2.069$

$E_{in} = E_0 \left(\frac{c}{\omega_{res} R}\right)^3.$

$g(\omega) = \max_{\vec{r}} \frac{|\vec{E}(\vec{r})|}{|\vec{E}_0|} = |\epsilon(\omega_{res})| \left(\frac{c}{\omega_{res} R}\right)^3 \approx 90$

$\Longrightarrow R \approx 36$ nm

Acknowledgment: We thank Dr. Vitaly Shevchenko for contributing this interesting problem.

END of Problem 2
问题 2 完