Pan Pearl River Delta Physics Olympiad 2021
2021 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助

Simplified Chinese Part-2 (Total 2 Problems, 60 Points)
简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 15 May 2021)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Quasicrystals (28 points)
问题 1: 准晶体 (28 分)

In 2011, the Nobel Prize in Chemistry was awarded to the discovery of quasicrystals. Nowadays, quasicrystals can be found in many applications such as the hardening of steel. How do quasicrystals differ from ordinary crystals?

2011 年，准晶体的发现者荣获诺贝尔化学奖。至今，我们已可见到准晶体不同的应用，例如钢的硬化。究竟准晶体与普通晶体有何不同？

In crystals, atoms are arranged in a periodic manner. The structure of crystals is known by periodically replicating the basic unit of the arrangement of a small number of atoms (Fig. 1 (Left)).

在晶体中，原子以周期性方式排列。通过周期性地复制少量原子排列的基本单位，就可以得到晶体的结构（图 1（左））。

On the other hand, atoms in quasicrystals are arranged in an orderly manner, but the local arrangement cannot be repeated by replication (Fig. 1(Right)).

另一方面，准晶体中的原子以有序的方式排列，但是局部的排列不能通过复制而得到全部结构（图 1（右））。

However, quasicrystals are far from random. They have a “hidden order”. For example, the structure in Fig. 1(Right) can be considered as a 5-dimensional cubic structure projected onto two dimensions. To understand this idea, we will consider a 1-dimensional quasicrystal projected from a 2-dimensional square lattice in this problem.

但是，准晶体远远不是随机的。他们有一个“隐藏的规律”。例如，图 1（右）中的结构可以考虑为投影到二维空间的 5 维立方结构。为了理解这个想法，我们在本题中将考虑从二维方格投影的一维准晶体。

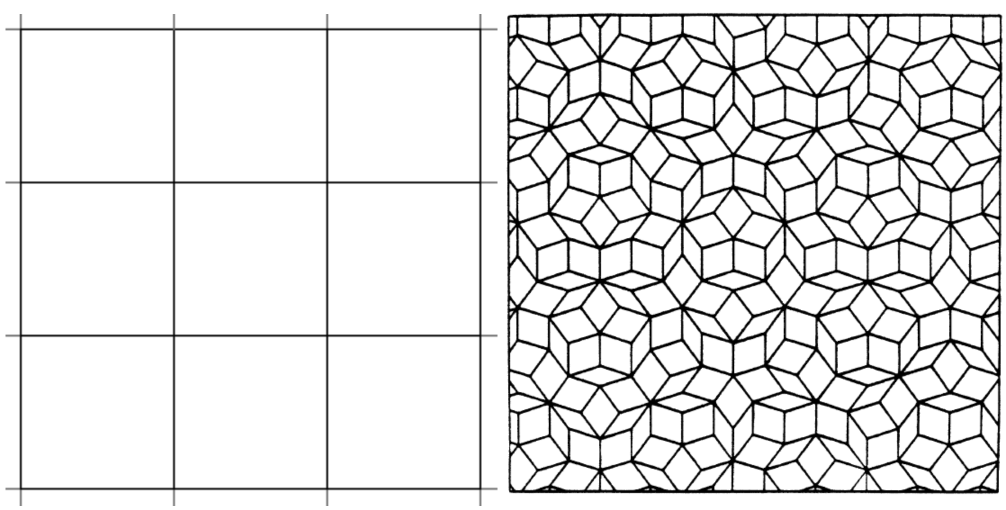


Fig. 1: (Left) A crystal (Right) A quasicrystal, known as Penrose tiling.
图 1：（左）晶体（右）称为 Penrose tiling 的准晶体。

A. Structure of Quasicrystals (14 points) 准晶体的结构（14 分）

Figure 2 shows a two-dimensional lattice in which the atoms are located at $(x_1, x_2) = (m_1 a, m_2 a)$ where m_1, m_2 are integers and a is the lattice spacing. In Section A, we assume $a = 1$. We construct a stripe defined by the condition

图 2 显示了一个二维晶格，其中原子位于$(x_1, x_2) = (m_1 a, m_2 a)$，$m_1$、$m_2$是整数，而$a$是晶格间距。在 A 部中，我们假设$a = 1$。 我们考虑一个条带，其定义为

$$\frac{x_1}{\tau} \leq x_2 < \frac{x_1}{\tau} + \tau.$$

τ is the irrational number $\tau = (1 + \sqrt{5})/2$. The inclination angle α of the strip is given by

τ 是无理数 $\tau = (1 + \sqrt{5})/2$。条带的倾角为

$$\alpha = \arctan \frac{1}{\tau}.$$

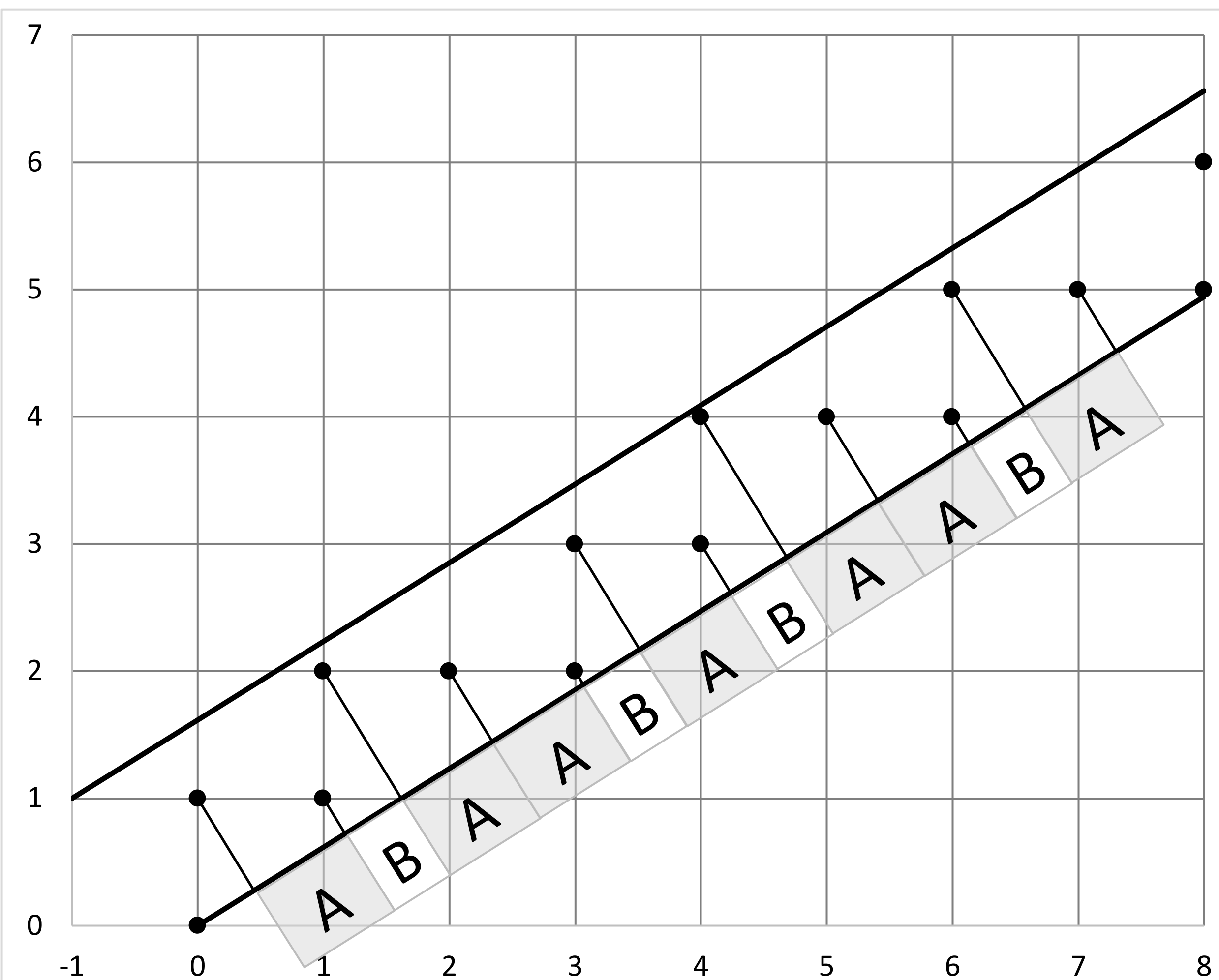


Fig. 2: Illustration of the projection method of obtaining a quasicrystal. 图 2：以投影法显示准晶体。

We project all lattice points lying within the strip to the line L_1 defined by $x_2 = x_1/\tau$. Since τ is an irrational number, the projected atomic positions on L_1 form a 1-dimensional quasicrystal. The lattice spacing now have two possible values.
我们将位于条带内的所有晶格点投影到L_1线上，L_1的定义为$x_2 = x_1/\tau$。由于τ是无理数，投影到L_1线上的原子位置便形成了一维准晶体。现在，晶格间距有两个可能的值。

A1	Calculate the lengths of the lattice spacings A and B. Write your answer as an expression containing τ. 计算晶格间距 A 和 B。答案以含τ的表达式写出。	2 points 2 分

$$A = \cos\alpha = \frac{\tau}{\sqrt{1+\tau^2}},$$
$$B = \sin\alpha = \frac{1}{\sqrt{1+\tau^2}}.$$

A2	Write the position $z(m_1, m_2)$ of the atom of the quasicrystal projected from (m_1, m_2) onto the line L_1 (that is, the displacement of the atom from (0,0)). Write your answer as an expression containing τ. 写下从(m_1, m_2)投影到L_1线上的准晶体原子位置$z(m_1, m_2)$（即从(0,0)到原子的位移）。答案以含τ的表达式写出。	1 point 1 分

$$z(m_1, m_2) = \frac{\tau m_1 + m_2}{\sqrt{1+\tau^2}}.$$

A3	Calculate the average lattice spacing d when the lattice length is very long. Write your answer as an expression containing τ. 计算当晶格长度很长时的平均晶格间距d。答案以含τ的表达式写出。	2 points 2 分

Consider the end point (m_1, m_2) of the lattice [1].
The length of the lattice approaches $\sqrt{m_1^2 + m_2^2}$.
The number of lattice points approaches $m_1 + m_2$
Furthermore, m_2 approaches m_1/τ.

$$d = \frac{\sqrt{m_1^2 + m_2^2}}{m_1 + m_2} = \frac{\sqrt{1 + 1/\tau^2}}{1 + 1/\tau} \text{ or } \frac{\sqrt{\tau^2+1}}{\tau+1} \text{ or } \frac{\sqrt{\tau^2+1}}{\tau^2}.$$

Note that the configuration of the lattice spacings A and B are no longer regular. It was observed that the configuration can be generated by the famous Fibonacci sequence, which uses the substitution rule
注意，晶格间距 A 和 B 的排列不再规则。我们观察到晶格的结构可以由著名的 Fibonacci sequence 生成，该序列使用替换规则

$$A \to AB, \quad B \to A.$$

For example, the configuration of the first seven spacings $ABAABABA$ in Fig. 2 is obtained by five substitutions starting from B.
例如，图 2 中前七个间距$ABAABABA$的排列是通过从 B 开始的五个替换获得的。

A4	Write the sequence after six substitutions starting from B. 写下从 B 开始六次替换后的序列。	1 point 1 分

$$ABAABABAABAAB$$

A5	Let n_A and n_B be the number of A and B in the sequence. Write the numbers n'_A and n'_B of A and B after one substitution. 令n_A和n_B为序列中A和B的数目。写下一次替换后，A 和 B 的数目n'_A 和n'_B。	2 points 2 分

$$n'_A = n_A + n_B,$$

$$n'_B = n_A.$$

A6	Calculate the ratio n_A/n_B when the sequence is very long. 计算当序列很长时的比例n_A/n_B。	2 points 2 分

Let $r = n_A/n_B$. Dividing the above equations,

$$r' = 1 + \frac{1}{r}.$$

When the chain is very long, r' approaches r. Hence,

$$r = 1 + \frac{1}{r}.$$

$$r^2 - r - 1 = 0.$$

$$r = \frac{1+\sqrt{5}}{2} = \tau.$$

For further analysis, we introduce two mathematical notations. For a real number x, $\lfloor x \rfloor$ denotes the integer part of x, and $\mathrm{frac}(x)$ denotes the fractional part of x. For example,

为了进一步分析，我们引入两个数学符号。对于实数x，$\lfloor x \rfloor$表示 x的整数部分，而$\mathrm{frac}(x)$则表示x的小数部分。例如，

$$\lfloor 1.73 \rfloor = 1 \quad \text{and} \quad \mathrm{frac}(x) = 0.73.$$

Labeling the atom at $(0,0)$ as $j = 0$, the position of the jth atom in the quasicrystal is given by

将$(0,0)$处的原子标记为 $j = 0$，准晶体中第j个原子的位置为

$$z_j = jd + \text{frac}\left(\frac{j}{\tau}\right)\Delta.$$

A7	Derive Δ as an expression containing τ. 推导Δ的表达式，式中含有τ。	2 points 2 分

Method 1:

We can see from Fig. 2 that when $j = 1$,

$$z_1 = \frac{1}{\sqrt{\tau^2+1}}.$$

$$\text{frac}\left(\frac{1}{\tau}\right) = \frac{1}{\tau}.$$

From A3,

$$d = \frac{\sqrt{\tau^2+1}}{\tau^2}.$$

Substituting into the equation,

$$\frac{1}{\sqrt{\tau^2+1}} = \frac{\sqrt{\tau^2+1}}{\tau^2} + \frac{\Delta}{\tau}.$$

$$\Delta = -\frac{1}{\tau\sqrt{\tau^2+1}}.$$

Method 2:

Note that (see the proof below)

$$m_1 = \left\lfloor\frac{j}{\tau}\right\rfloor,$$

$$m_2 = j - m_1 = j - \left\lfloor\frac{j}{\tau}\right\rfloor.$$

$$z_j = \frac{1}{\sqrt{1+\tau^2}}\left[\tau\left\lfloor\frac{j}{\tau}\right\rfloor + j - \left\lfloor\frac{j}{\tau}\right\rfloor\right] = \frac{j}{\sqrt{1+\tau^2}} + \frac{\tau-1}{\sqrt{1+\tau^2}}\left[\frac{j}{\tau} - \text{frac}\left(\frac{j}{\tau}\right)\right]$$

$$= \frac{j}{\sqrt{1+\tau^2}} + \frac{j}{\tau^2\sqrt{1+\tau^2}} - \frac{\tau-1}{\sqrt{1+\tau^2}}\text{frac}\left(\frac{j}{\tau}\right) = \frac{\sqrt{1+\tau^2}}{\tau^2}j - \frac{1}{\tau\sqrt{1+\tau^2}}\text{frac}\left(\frac{j}{\tau}\right).$$

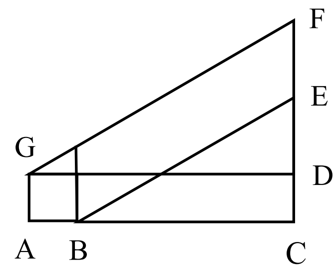
$$\Delta = -\frac{1}{\tau\sqrt{\tau^2+1}}.$$

Proof of $m_1 = \lfloor j/\tau \rfloor$:
Noting that the upper boundary passes through the point $(-1,1)$, we can calculate the number of atoms enclosed in the strip by subtracting the number of atoms in the area BCE from ACFG.

$$\text{Number of atoms in area BCE} = \sum_{i=0}^{m_1} \left(\left\lfloor \frac{i}{\tau} \right\rfloor + 1 \right).$$

$$\text{Number of atoms in area GDF} = \sum_{i=0}^{m_1+1} \left(\left\lfloor \frac{i}{\tau} \right\rfloor + 1 \right).$$

$$\text{Number of atoms in area ACFG} = \sum_{i=0}^{m_1+1} \left(\left\lfloor \frac{i}{\tau} \right\rfloor + 1 \right) + m_1 + 2.$$



$$\text{Number of atoms in the strip} + \text{atom A} + \text{atom G} + \sum_{i=0}^{m_1} \left(\left\lfloor \frac{i}{\tau} \right\rfloor + 1 \right) - \text{atom B} = \sum_{i=0}^{m_1+1} \left(\left\lfloor \frac{i}{\tau} \right\rfloor + 1 \right) + m_1 + 2.$$

$$\text{Number of atoms in the strip} (= j + 1) = \sum_{i=0}^{m_1+1} \left(\left\lfloor \frac{i}{\tau} \right\rfloor + 1 \right) + m_1 + 1 - \sum_{i=0}^{m_1} \left(\left\lfloor \frac{i}{\tau} \right\rfloor + 1 \right)$$

$$= \left\lfloor \frac{m_1 + 1}{\tau} \right\rfloor + m_1 + 2.$$

$$j = \left\lfloor \frac{m_1 + 1}{\tau} \right\rfloor + m_1 + 1.$$

For each value of m_1, there are at most two atoms. For the atom with a higher value of m_2,

$$j = \frac{m_1 + 1}{\tau} + (m_1 + 1) - \operatorname{frac}\left(\frac{m_1 + 1}{\tau} \right) = (m_1 + 1)\tau - \operatorname{frac}\left(\frac{m_1 + 1}{\tau} \right).$$

$$\frac{j}{\tau} = m_1 + 1 - \frac{1}{\tau} \operatorname{frac}\left(\frac{m_1 + 1}{\tau} \right),$$

$$m_1 < \frac{j}{\tau} < m_1 + 1,$$

$$m_1 = \left\lfloor \frac{j}{\tau} \right\rfloor.$$

For the atom with a lower value of m_2, we note that its previous atom belongs to the group with $m_1 - 1$. Hence

$$j = \left(\left\lfloor \frac{m_1}{\tau} \right\rfloor + m_1 \right) + 1 = \frac{m_1}{\tau} + m_1 + 1 - \operatorname{frac}\left(\frac{m_1}{\tau} \right) = m_1 \tau + 1 - \operatorname{frac}\left(\frac{m_1}{\tau} \right).$$

$$\frac{j}{\tau} = m_1 + \frac{1}{\tau} - \frac{1}{\tau} \operatorname{frac}\left(\frac{m_1}{\tau} \right).$$

$$m_1 < \frac{j}{\tau} < m_1 + 1,$$

$$m_1 = \left\lfloor \frac{j}{\tau} \right\rfloor.$$

A8	Calculate the coordinate (m_1, m_2) of the atom with $j = 101$. 计算$j = 101$的原子的坐标(m_1, m_2)。	2 points 2 分

$$\operatorname{frac}\left(\frac{101}{\tau} \right) = \frac{101}{\tau} - \left\lfloor \frac{101}{\tau} \right\rfloor = \frac{101}{\tau} - 62.$$

$$z_{101} = 101\frac{\sqrt{\tau^2+1}}{\tau^2} - \frac{1}{\tau\sqrt{\tau^2+1}}\left(\frac{101}{\tau} - 62\right) = \frac{101}{\sqrt{\tau^2+1}} + \frac{62(\tau-1)}{\sqrt{\tau^2+1}} = \frac{39+62\tau}{\sqrt{\tau^2+1}}.$$

On the other hand,

$$z_j = m_1 A + m_2 B = \frac{\tau m_1 + m_2}{\sqrt{\tau^2+1}}.$$

Hence, $(m_1, m_2) = (62, 39)$.

B. Diffraction Pattern of Quasicrystals (14 points) 准晶体的衍射图案（14 分）

Quasicrystals were first discovered by observing their characteristic diffraction pattern.

准晶体是首先通过观察其特征衍射图案来发现的。

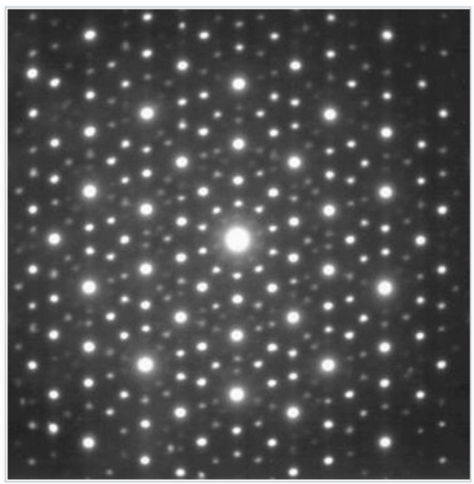


Fig. 3: Electron diffraction pattern of a quasicrystal. 图 3：准晶体的电子衍射图。

Figure 3 shows the electron diffraction pattern of a quasicrystal. A crystal is said to have n-fold symmetry if its diffraction pattern is identical if it is rotated by an angle of $2\pi/n$.

图 3 显示一种准晶体的电子衍射图。如果晶体旋转角度为$2\pi/n$后，其衍射图案相同，则该晶体具有n倍对称性。

B1	Identify the symmetries of the diffraction pattern in Fig. 3. 辨认图 3 衍射图案的对称性。	2 points 2 分

The diffraction pattern in Fig. 3 has 2-fold, 5-fold and 10-fold symmetries [2]. Note that since pentagonal structures cannot fill up the space fully, they cannot form periodic structures. See Penrose tiling in Fig. 1. The 5-fold and 10-fold symmetries are characteristics of quasicrystals.

To understand how the diffraction pattern can be derived from the projection method, we first consider the diffraction pattern of a 1-dimensional crystal.

为了了解如何从投影法中得出衍射图案，我们首先考虑一维晶体的衍射图案。

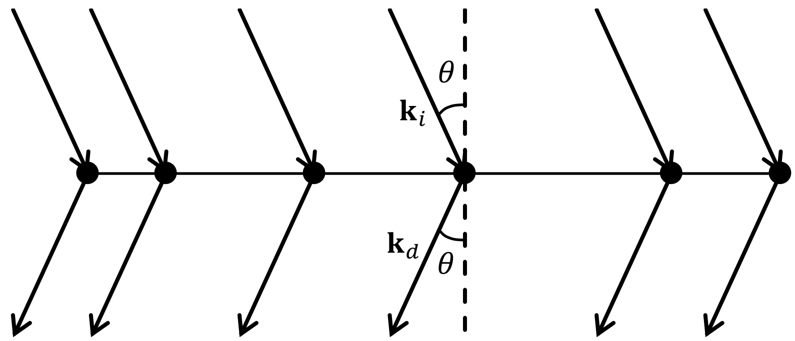


Fig. 4: Light diffraction by a lattice. 图 3：晶格的光衍射。

As shown in Fig. 4, the 1-dimensional lattice consists of N atoms. The position of atom j is x_j. A light wave with wavevector $\mathbf{k}_i$ is incident on the lattice at an angle θ with the normal direction and is diffracted at the same angle with the normal. The diffracted wave has a wavevector $\mathbf{k}_d$ with the same magnitude as $\mathbf{k}_i$. The change in the wavevector is denoted as

如图 3 所示，一维晶格由N个原子组成。 原子j的位置是x_j。光波入射到晶格中，入射波矢量为$\mathbf{k}_i$，与法线成角度θ，衍射后方向与法线成相同的角度。衍射波的波矢$\mathbf{k}_d$ 的大小与$\mathbf{k}_i$相同。 波矢的变化表示为

$$\mathbf{q} = \mathbf{k}_d - \mathbf{k}_i.$$

The magnitudes of the wavevectors are denoted as $|\mathbf{k}_d| = |\mathbf{k}_i| = k$ and $|\mathbf{q}| = q$. Note that q is a monotonic function of the diffraction angle θ, and so can represent the diffraction direction.

波矢的大小表示为$|\mathbf{k}_d| = |\mathbf{k}_i| = k$和$|\mathbf{q}| = q$。注意，$q$是衍射角$\theta$的单调函数, 因此可以表示衍射方向。

B2	Write the expression of q as a function of the diffraction angle θ. 写下q作为衍射角θ函数的表达式。	1 point 1 分

$$q = 2k \sin\theta.$$

The positions and magnitudes of the diffraction peaks are given by the structure factor defined as
衍射峰的位置和大小由结构因子给出，其定义为

$$S(\mathbf{q}) = \frac{1}{N}\left[\sum_{j=1}^{N} \cos(q_x x_j)\right]^2 + \frac{1}{N}\left[\sum_{j=1}^{N} \sin(q_x x_j)\right]^2,$$

where q_x is the x component of $\mathbf{q}$. 其中q_x是$\mathbf{q}$的x分量。

Remark: Students who are familiar with complex numbers may use the definition

备注：熟悉复数的同学可以使用下列定义

$$S(\mathbf{q}) = \frac{1}{N}\left|\sum_{j=1}^{N} \exp(iq_x x_j)\right|^2.$$

Consider a 1-dimensional periodic lattice in which $x_j = jd$. 考虑一维周期晶格，其中$x_j = jd$。

B3	What are the values of q at the peak positions of the diffraction pattern of the periodic lattice? 在周期晶格的衍射图案中，峰值位置处的q值是多少？	2 points 2 分

When $qd = 2n\pi$ for integer n,

$$S(\mathbf{q}) = \frac{1}{N}\left[\sum_{j=1}^{N} \cos(qjd)\right]^2 + \frac{1}{N}\left[\sum_{j=1}^{N} \sin(qjd)\right]^2 = \frac{1}{N}\left[\sum_{j=1}^{N} 1\right]^2 + \frac{1}{N}\left[\sum_{j=1}^{N} 0\right]^2 = N.$$

Hence, the diffraction peaks are located at $q = \frac{2n\pi}{d}$ where n is an integer.

For students using complex numbers, when $qd = 2n\pi$ for integer n,

$$S(\mathbf{q}) = \frac{1}{N}\left|\sum_{j=1}^{N} e^{iqjd}\right|^2 = \frac{1}{N}\left|\sum_{j=1}^{N} e^{i2\pi nj}\right|^2 = \frac{1}{N}\left[\sum_{j=1}^{N} 1\right]^2 = N.$$

Now consider the case that each atom in the periodic lattice is "smeared" out to a length b, where $b < d$. This means that the density $\rho(x)$ of the lattice becomes

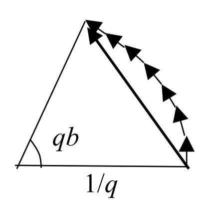
现在考虑以下情况：周期晶格中的每个原子都被"抹散"到长度b，其中$b < d$。这意味着晶格的密度$\rho(x)$变为

$$\rho(x) = \begin{cases} \frac{1}{b} & jd \le x \le jd + b, \\ 0 & \text{otherwise.} \end{cases}$$

The diffraction peaks do not have the same magnitude any longer. 衍射峰不再具有相同的大小。

B4	Calculate the magnitudes of the diffraction peaks at $\mathbf{q}$ of the smeared lattice. 计算抹散晶格$\mathbf{q}$处衍射峰的大小。	2 points 2 分

The structure factor can be calculated by the phasor method. As shown in the figure, the length of each element vector is dx, and it spans an angle qdx. Hence, the radius of the arc is $dx/qdx = 1/q$. The total angle of rotation is qb. Hence, the magnitude of the vector sum is $2(1/q)\sin qb/2$.



$$S(\mathbf{q})=\frac{1}{N}\left(\frac{N}{b}\frac{2}{q}\sin\frac{qb}{2}\right)^2=N\left(\frac{\sin\frac{qb}{2}}{\frac{qb}{2}}\right)^2.$$

For students using complex numbers,

$$S(\mathbf{q})=\frac{1}{N}\left|\frac{N}{b}\int_0^b dx\, e^{iqx}\right|^2=N\left|\frac{e^{iqb}-1}{iqb}\right|^2=N\left(\frac{\sin\frac{qb}{2}}{\frac{qb}{2}}\right)^2.$$

Now consider the quasicrystal in Part A. Since there are two incommensurate lattice spacings in the quasicrystal, its diffraction peaks are given by wavevectors with a pair of indices,

现在考虑 A 部中的准晶体。由于在准晶体中存在两种不相称的晶格间距，因此准晶体的衍射峰由具有一对指数的波矢给出，

$$q_{mn}=\frac{2\pi}{d}\left(m+\frac{n}{\tau}\right).$$

B5	Consider the phase $\phi=q_{mn}z_j$ in the structure factor of quasicrystals. Write the expression of the phase in the form 考虑准晶体中结构因子的相$\phi=q_{mn}z_j$。试写下相的表达式 $$\phi=2\pi F+X\text{frac}\left(\frac{j}{\tau}\right),$$ where F is an integer and X is a real number. 其中F是整数，X是实数。	2 points 2 分

$$\phi=q_{mn}z_j=\frac{2\pi}{d}\left(m+\frac{n}{\tau}\right)\left[jd+\Delta\text{frac}\left(\frac{j}{\tau}\right)\right]$$

$$=2\pi\left(mj+n\left\lfloor\frac{j}{\tau}\right\rfloor\right)+2\pi\left[m\frac{\Delta}{d}+n\left(1+\frac{\Delta}{\tau d}\right)\right]\text{frac}\left(\frac{j}{\tau}\right)$$

$$=2\pi\left(mj+n\left\lfloor\frac{j}{\tau}\right\rfloor\right)+\frac{2\pi\tau(n\tau-m)}{\tau^2+1}\text{frac}\left(\frac{j}{\tau}\right).$$

Hence,

$$F=mj+n\left\lfloor\frac{j}{\tau}\right\rfloor.$$

$$X=2\pi\left[m\frac{\Delta}{d}+n\left(1+\frac{\Delta}{\tau d}\right)\right]\quad\text{or}\quad\frac{2\pi\tau(n\tau-m)}{1+\tau^2}.$$

B6	Calculate the magnitude of the diffraction peak at q_{mn}. 计算在q_{mn}处的衍射峰的大小。	2 points 2 分

In the expression of ϕ, the first term is an integer multiple of 2π, and therefore contributes to a factor of unity upon exponentiation. In the second term, the value is uniformly and densely distributed in the interval $(0, X)$ as j covers an infinite range and τ is an irrational number. Hence, the structure factor is given by [3]

$$S(q_{mn}) = \frac{1}{N}\left|\frac{N}{X}\int_0^X dy\, e^{iy}\right| = N\left(\frac{\sin\frac{X}{2}}{\frac{X}{2}}\right)^2.$$

The same solution can be obtained by the phasor method.

B7	Find (m, n) for the highest diffraction peak in the range $0 \le n \le 3$, excluding $(m, n) = (0,0)$. Then calculate q_{mn} (in units of $2\pi/d$) and the magnitude of this peak. 找出在范围$0 \le n \le 3$内最高衍射峰(m, n)，不包括$(m, n) = (0,0)$。然后计算q_{mn}（以$2\pi/d$为单位）和该峰值的大小。	3 points 3 分

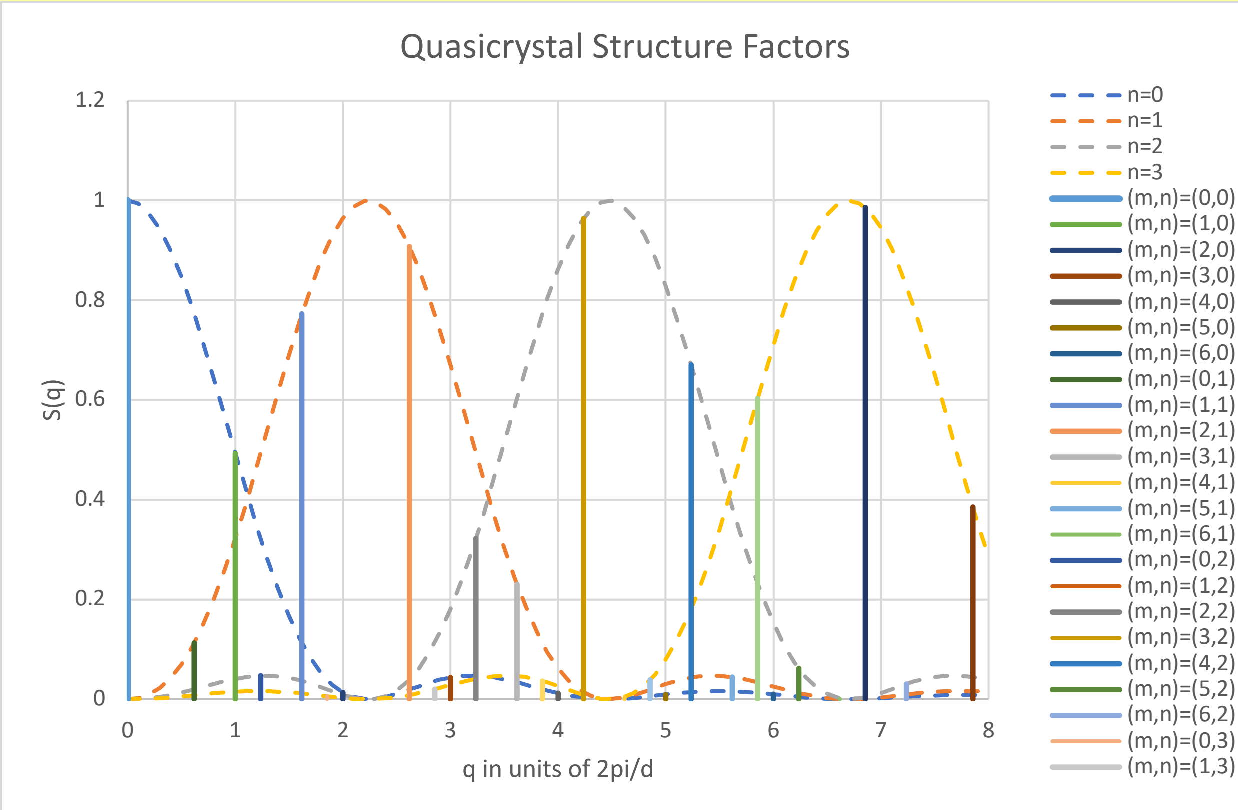
The diffraction peak is highest when X is smallest, which in turn means m/n is closest to $\tau = 1.618$. In the range $0 \le n \le 3$, this is given by $(m, n) = (5,3)$ since $5/3 = 1.667$.

$$q_{mn} = \frac{2\pi}{d}\left(m + \frac{n}{\tau}\right) = 6.854\frac{2\pi}{d}.$$

$$X = \frac{2\pi\tau(n\tau - m)}{1 + \tau^2} = -0.4100$$

$$S(q_{mn}) = N\left(\frac{\sin\frac{X}{2}}{\frac{X}{2}}\right)^2 = 0.9861N.$$

The diffraction peaks in the range $0 \le n \le 3$ are plotted in the following figure. Although the ordering of the diffraction peaks is difficult to follow, envelopes for each value of n between 0 and 3 are plotted to illustrate the hidden conditions.



References:

[1] A. N. Poddubny and E. L. Ivchenko, "Photonic Quasicrystalline and Aperiodic Structures", Physica E **42** (2010) 1871-1895.

[2] Figure 3 is reproduced from:
Tsutomu Ishimasa, Shiro Kashimoto and Ryo Maezawa, "Search and Synthesis of New Family of Quasicrystals", MRS Online Proceedings Library (OPL) , Volume 805: Symposium LL – Quasicrystals , 2003 , LL1.1

[3] Dov Levine and Paul J. Steinhardt, "Quasicrystals. I. Definition and structure", Phys. Rev. B **34**, 596-616 (1986).

Problem 2: Black Hole Physics (32 points)
问题 2: 黑洞物理 (32 分)

Black holes are the most mysterious objects in our universe. A black hole is surrounded by an event horizon (horizon for short). Anything that falls through the horizon into the black hole cannot escape. If a black hole is stationary, not rotating, has no electric charge, then the horizon is spherical, with radius $R = 2GE/c^4$, which can be also written as $R = 2GM/c^2$ using Einstein's energy-mass relation $E = Mc^2$. Here G, c, M and E are the Newton's constant, speed of light, and mass and energy of the black hole, respectively. The horizon area is thus $A = 4\pi R^2$. The singularity "inside" the black hole is one of the greatest mysteries in the theory of gravity, since the energy density of the singularity appears to diverge and the classical general relativity fails to operate there.

黑洞是宇宙中最神秘的天体。黑洞被事件视界 (简称视界) 环绕。通过视界掉进黑洞的东西不能逃出来。如果黑洞是稳态的，不旋转也不带电，则黑洞视界是球形的，其半径为 $R = 2GE/c^4$，利用爱因斯坦的质能关系 $E = Mc^2$，视界半径也可以写作 $R = 2GM/c^2$ 。这里 G, c, M 和 E 分别为牛顿引力常数、光速、黑洞质量和黑洞的能量。视界面积是 $A = 4\pi R^2$。黑洞内部的奇点是最神秘的引力现象之一，在奇点附近，能量密度趋于无穷大，经典广义相对论不再适用。

In the following, we will discuss the formation, thermodynamics and rotation of black holes, and how a civilization may use black holes as power plants.

在本题中，我们将讨论黑洞的形成、热力学和旋转黑洞，以及从黑洞提取能量的可能性。

Note: To avoid the usage of general relativity, in this problem, no concepts about curved spacetime will be introduced. You do not need to think about curved spacetime when working on this problem.

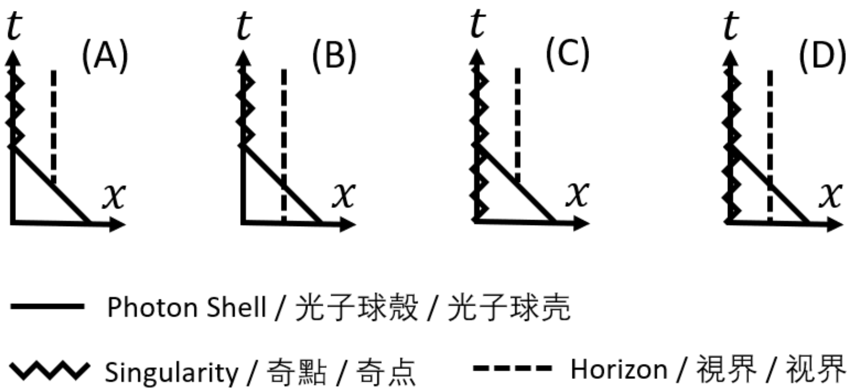
注：为避免使用广义相对论知识，本题将不会涉及时空弯曲等概念。答题时不需要考虑时空弯曲。

PART A. FORMATION AND PROPERTIES 黑洞的形成和性质

We study a simple formation mechanism of black holes. Consider a spherical shell of photons (quanta of light) is moving towards the center of the shell to form a black hole. The self-interaction of the photons can be ignored.

我们研究一个简单的黑洞形成机制。考虑一个光子 (光的量子) 球壳。这个球壳中的光子向球壳中心运动，以形成黑洞。我们忽略光子之间的自相互作用。

A1	Assume that the wavelength of photons is short enough, and thus the shell is thin (this short wavelength assumption only applies for this Question A1, and may not apply for later questions), which of the following describes the formation of the horizon and the singularity of a black hole? 假设光子波长足够短，所以球壳很薄 (这个短波长近似只用在本小题，即 A1 中，后面的题中我们不再假设短波长近似)。下列哪一个图像描述了黑洞视界和奇点的形成？	1 point 1 分



Solution: A

A2	Suppose the photons in the shell all have wavelength λ. Thus, the energy of each photon is $E_\gamma = hc/\lambda$, where h is the Planck's constant. To make a black hole with horizon radius R, what is the number of photons N needed? 假设球壳中所有光子的波长都是 λ。所以，每个光子的能量为 $E_\gamma = hc/\lambda$，其中 h 是普朗克常数。为了形成视界半径为 R 的黑洞，求所需的光子数 N。	1 point 1 分

Solution: $N = \frac{E}{E_\gamma} = \frac{c^3 R\lambda}{2Gh}$

Entropy of infalling photons 坍缩过程中光子的熵

A3	For a shell of N photons, the entropy can be written as $S_{N\gamma} = \alpha\, k_B^{c_1} N^{c_2}$. Here k_B is the Boltzmann's constant and α is a dimensionless constant of order 1. (In the below analytical formulae, α should be kept explicitly. In order of magnitude estimations, α can be set to 1.) Find integer numbers c_1 and c_2. N 个光子组成的球壳的熵可以写成 $S_{N\gamma} = \alpha\, k_B^{c_1} N^{c_2}$。这里 k_B 是玻尔兹曼常数，$\alpha \sim O(1)$ 是一个无量纲常数。(在下面各题的解析公式中，请保留α。在数量级估计中，可以将 α 的值取为 1). 求整数 c_1 和 c_2。	1 point 1 分

Solution: $c_1 = 1$ from dimensional analysis; $c_2 = 1$ from that entropy of photons is an extensive quantity.

Entropy of the black hole 黑洞的熵

A4	Taking $\lambda = R$, we have the entropy of the black hole $S = S_{N\gamma}$. (If $\lambda > R$, the photons are too non-local to form a black hole. Thus, the $\lambda = R$ photons carry the largest amount of information.) Write S in terms of R, α and the constants of Nature. 设 $\lambda = R$，我们得到黑洞的熵 $S = S_{N\gamma}$。(这是因为，如果 $\lambda > R$，则光子延展范围太大，不能坍缩成黑洞。所以，$\lambda = R$ 光子可以携带最多的信息。) 请用 R, α 和基本物理常数表示 S。	1 point 1 分

Solution: $S = \alpha\, k_B N = \frac{\alpha\, k_B\, R^2 c^3}{2Gh}$.

A5	Based on the black hole entropy formula, which of the physical observation is *incorrect*? Choose one from the below answers. A. Black hole entropy is an interdisciplinary research direction of thermal, quantum and gravitational physics. B. The black hole entropy is an extensive quantity which scales as the volume of the black hole. C. The existence of black entropy indicates that black hole should contain many microstates. D. Black holes are gravitational systems with non-perturbative quantum effects, and are thus a key to quantum gravity. 根据黑洞熵的公式，下面哪项是错误的？请只选择一项。 A. 黑洞熵是热力学、量子理论和引力的交叉学科。 B. 黑洞熵是广延量，与黑洞的体积成正比。 C. 黑洞存在熵，所以也应该存在微观状态。 D. 黑洞是具有非微扰量子特征的引力系统，所以是通向量子引力的一把钥匙。	1 point 1 分

Solution: B (BH entropy scales as area, not volume, which is very special from other entropies.)

A is correct because the formula contains k_B (thermal), h (quantum), c (special relativity) and G (general relativity)

C is correct as a usual interpretation of entropy indicates.

D is correct because h is in the denominator.

Part B. Entropy bounds of nature 黑洞熵限

It is conjected that black holes are the densest objects of nature, not only in energy, but also in entropy and information. Based on this observation, estimate the quantities in Part B. By estimate, you need to get the correct order of magnitude. Order one coefficients may be neglected.

物理学家猜测，黑洞不仅是世界上能量密度最大的物体，也是熵密度、信息密度最大的物体。基于这个猜测，请对 Part B 中的物理量做数量级估计。你只需要估计正确的数量级。$O(1)$ 的常数可以忽略。

B1	Nowadays, computer hard disks store information with a day-by-day increasing information density. However, to store information, enough number of states, and thus enough entropy is needed. This is understood from Boltzmann's statistical interpretation of entropy: $S = k_B \ln \Omega$, where Ω is the possible number of states of the	2 points 2 分

	system. Consider a spherical hard disk in the vacuum, with capacity $1\text{Tb} = 10^{12}$ bit. What is the minimal radius of this hard disk? 目前，电脑硬盘可以存储的信息密度越来越大。但是，为了存储信息，我们需要足够多的状态数，所以需要足够多的熵。这可以从玻尔兹曼熵的统计解释中看出：$S = k_B \ln\Omega$，其中 Ω 是系统可能处于的状态的数量。考虑一个真空中的球形硬盘，容量为 $1\text{Tb} = 10^{12}$比特。求硬盘的最小半径。	

Solution:

$\frac{S}{k_B} = \ln\Omega = 1Tb = 10^{12}.$

(Note: $\ln\Omega = 1Tb$, instead of $\Omega = 1Tb$, since 1Tb hard disk can represent 2^{1Tb} states. Here log base 2 or base e is ignored since it's order of magnitude estimate.)

$R = \sqrt{2Gh\, 10^{12}/c^3} = 6 \times 10^{-29}\text{m}$

The Bekenstein Entropy bound 贝肯斯坦熵限

B2	Consider a clump of matter with mass M_m (and thus energy E_m) and radius R_m. When this clump of matter falls into a black hole (the black hole has existed before the clump of matter falls in), we require that R_m should be not greater than the horizon radius of the initial black hole, to make sure this clump of matter can fall in. Denote the entropy of this clump of matter as S_m. Find a universal upper bound of S_m, in terms of E_m and R_m, but independent of parameters of the black hole, or Newton's gravitational constant G. 考虑质量为 M_m (所以能量为 E_m)，半径为 R_m 的一块物质。当这块物质掉进一个黑洞时 (物质掉落前，黑洞就已经存在了)，我们要求 R_m 不大于黑洞本来的视界半径。因为这样才能确保这块物质掉进去。利用这个过程，求这块物质的熵 S_m 的普适上限。你导出的 S_m 的上限需要用 E_m 和 R_m 表示，但不依赖于黑洞的参数，也不依赖于牛顿引力常数 G。 Note: Necessary steps of derivation is required. 注：需要写出必要的推导步骤。	4 points 4 分

Solution: We use subscript i to denote quantities of the initial black hole, subscript f to denote quantities of the final black hole after matter falls in. Thus, the conservation of energy and non-decrease of entropy reads:

$$E_f = E_i + E_m \quad \Rightarrow \quad E_m = \frac{c^4}{2G}(R_f - R_i).$$

$$S_f \geq S_i + S_m$$

Thus, $S_m \leq \frac{\alpha k_B c^3}{2Gh}(R_f - R_i)(R_f + R_i) = \frac{\alpha k_B}{hc} E_m (2R_i + \frac{2G}{c^4} E_m)$

In addition, we have $R_i \geq R_m$. For each possible initial black hole R_i, there is a corresponding bound. We should take the tightest bound in all this bounds by taking $R_i = R_m$ (note: this is not directly inserting $R_i \geq R_m$ to the above equation, because the direction of the inequality sign is different.) Thus,

$S_m \leq \frac{\alpha k_B}{hc} E_m \left(2R_m + \frac{2G}{c^4} E_m\right)$

Further, as we mentioned, black holes are densest objects in energy. For the clump of matter to be at most as dense as black holes, we have $\frac{2GE_m}{c^4} \leq R_m$. Thus,

$S_m \leq \frac{3\alpha k_B}{hc} E_m R_m$

B3	For a 1Tb hard disk with 1nm radius, what is the minimal mass of the hard disk? 对于一个半径为 1nm，容量为 1Tb 的硬盘，硬盘的质量至少为多大？	1 point 1 分

Solution: $10^{12}/(\frac{3c}{h} \times 10^{-9}\text{m})$ = 7×10^{-22}kg

Part C. Black hole Temperature and radiation 黑洞的温度和辐射

C1	Find the black hole temperature T in terms of horizon radius R. 求黑洞的温度 T，用视界半径 R 表示。	2 points 2 分

Solution: We already know the black hole energy and entropy. From the first law of thermodynamics,

$T = \frac{dE}{dS} = \frac{hc}{2\alpha\, k_B R}$.

HAWKING RADIATION 霍金辐射

C2	According to the Stefan-Boltzmann law, an object with a temperature T should radiate. Calculate the radiation power P in terms of the horizon radius R. 由斯蒂凡-玻尔兹曼定律，具有温度 T 的物体会发出辐射。计算辐射的功率 P，用视界半径 R 表示。 Note: the Stefan-Boltzmann constant σ can be written in more fundamental quantities as $\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$. 注：斯蒂凡-玻尔兹曼常数 σ 可以由更基本的物理量表达：$\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$。	2 points 2 分

Solution: $P = 4\pi R^2 \sigma T^4 = hc^2\pi^6/(30\alpha^4 R^2)$.

C3	Primordial black holes are a conjected type of black holes, which has existed almost from the "born" of the universe till now. Denote the mass of the primordial black hole by M_P when it has just formed in the primordial universe. For the primordial black holes that still exist now, estimate a lower bound for their M_P (ignore the accretion of the primordial black holes). 原初黑洞猜想认为，在宇宙诞生之初，就可能已经存在着一些黑洞。它们直到目前仍然存在。设 M_P 为在宇宙早期，原初黑洞刚形成时的质量。为了让原初黑洞直到现在仍然存在，求 M_P 的下限 (忽略原初黑洞的吸积)。	4 points 4 分

Solution:

There is a mass bound of primordial black holes, because if the primordial black holes are too light, then these black holes energy are all radiated away by Hawking radiation during the history of the universe. Thus, we consider the black holes which are just Hawking-radiated to zero mass now. They are the lightest primordial black holes and we calculate their M_P.

$$\frac{dM}{dt} = \frac{P}{c^2} = -\frac{h\pi^6 c^4}{120\alpha^4 G^2 M^2}$$

$M(t) = \left(\frac{h\pi^6 c^4}{40\alpha^4 G^2}(t_0 - t)\right)^{1/3}$, where $t_0 \sim 10^{10}$ years is the age of the universe (which we hope that you know).

At $t = 0$, $M_P > M(0) = 1.4 \times 10^{13}$kg.

PART D. ROTATING BLACK HOLES 旋转的黑洞

Realistic black holes are typically rotating, due to the angular momentum conservation of in-falling matter. With rotation, the first law of thermodynamics of a black hole is $dE = TdS + \Omega dJ$, where Ω can be understood as the angular velocity of the horizon, and J the angular momentum of the black hole. In the following, we consider the $\Omega \geq 0$ parameter regime.

现实世界中的黑洞一般是旋转的。这是因为坍缩成黑洞的物体一般携带角动量，以及角动量守恒。对于转动的黑洞，黑洞的热力学第一定律为$dE = TdS + \Omega dJ$，其中 Ω 可被理解为视界的角速度，J 是黑洞的角动量。在下题中，我们考虑 $\Omega \geq 0$ 的参数区间。

D1	Now, we let the black hole to interact with a clump of matter outside the black hole. After non-adiabatic interaction, part of the matter falls into the black hole, such that the change of energy and the change of angular momentum of the black hole satisfies $dE = \lambda dJ$, where λ is a constant. Find the range for λ for the black hole to lose energy after the interaction. 现在考虑黑洞与黑洞外面的物质相互作用。经过非绝热的相互作用，部分物质掉进黑洞。这个过程中，黑洞能量和角动量的变化满足 $dE = \lambda dJ$，其中 λ 是一个常数。为了让黑洞在与物质的相互作用中能量减小，求 λ 的取值范围。	2 points 2 分

Solution: Insert it to the first law, we have

$dE = \frac{TdS}{(1-\Omega/\lambda)}$. From $dS > 0$ (non-adiabaticity), the condition $dE < 0$ implies $0 < \lambda < \Omega$.

In D1 we have found out the principles of extracting energy from black holes. In practice, we study an explicit toy model of how matter extract energy from a toy "black hole" in Newtonian mechanics (i.e. no special relativity or general relativity needs to be considered). This is a simplified version of the so-called Penrose process.

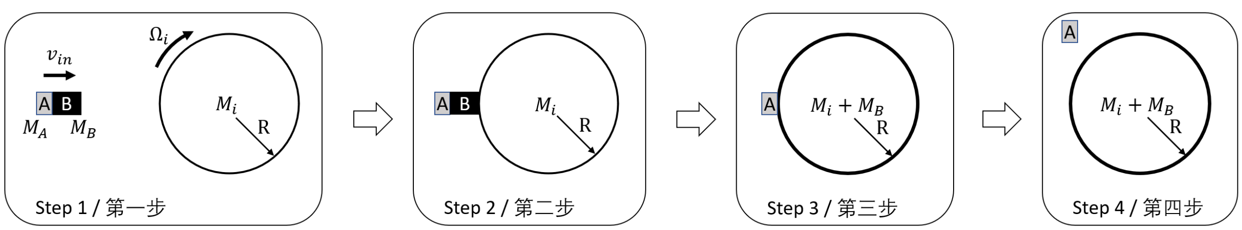
在 D1 中，我们发现了从黑洞中提取能量的一般原则。现在，我们研究一个玩具模型，来进一步理解物质如何从一个牛顿力学中的"玩具黑洞"（也就是说，不需要考虑狭义相对论和广义相对论效应）提取能量。这是"彭罗斯过程"的一个简化版本。

Let's model the rotating black hole as a rotating sticky ring with radius R, initially with mass M_i uniformly distributed on the ring, and angular velocity Ω_i. Its center of mass is initially at rest. We neglect the gravitational effects of this ring (i.e. the ring does not source a gravitational force in our approximation).

让我们用一个半径为 R，旋转的，有粘性的环来模拟黑洞。环的初始质量为 M_i (均匀分布于环上)，角速度为 Ω_i。在初始时刻，环的质心是静止的。我们忽略环的引力效应 (也就是说，在我们的近似下，这个环并不产生引力)。

Now, consider a composite particle AB (there is a force to bind A and B, but the binding energy is negligible), where part A and part B has mass M_A and M_B, respectively. A and B are considered as point mass.

现在，考虑一个复合粒子 AB (A 和 B 之间的力把 AB 束缚在一起，但是束缚 AB 的势能可以忽略)，其中 A 部分和 B 部分分别具有质量 M_A 和 M_B。A 和 B 都可以看成质点。



Step 1: the composite particle AB moves toward the center of the black hole with an initial velocity v_{in}.

第一步：复合粒子以初始速度 v_{in} 朝着环的中心运动。

Step 2: AB stick on the black hole surface, and rotate together with the black hole.

第二步：AB 粘在黑洞表面上，并且随着黑洞转动

Step 3: B got absorbed by the black hole. To simplify the calculation, *assume* here (in D2 and D3) that after absorbing B, the black hole is still a uniform ring with radius R, and its new mass is $M_i + M_B$.

第三步：B 被黑洞吸收。为了简化计算，假设 (在 D2 和 D3 题中) B 被吸收后，黑洞仍然用一个均匀圆环表示，半径仍为 R，而环的质量变成了 $M_i + M_B$。

Step 4: At the moment B got completely absorbed, the binding between B and A disappear, and then A moves freely to the tangent direction of the black hole in the black hole reference frame.

第四步：当 B 完全被吸收的瞬间，B 和 A 之间的束缚消失了。于是，在黑洞的参考系下，A 沿着旋转的切线方向自由飞出。

Steps 2, 3, 4 happen fast enough, such that the amount of rotation of the ring during these steps is negligible.

第二、三、四步发生得足够快，这些步骤中圆环转过的角度可以忽略不计。

D2	Find the condition that the ejected kinetic energy K_{out} for particle A is greater than the initial incoming kinetic energy for the composite particle $K_{in} = \frac{1}{2}(M_A + M_B)v_{in}^2$, in the form of $\Omega_i > \cdots$ or $\Omega_i < \cdots$. 设粒子 A 的出射动能为 K_{out}，复合粒子的初始动能为 $K_{in} = \frac{1}{2}(M_A + M_B)v_{in}^2$。求 $K_{out} > K_{in}$ 的条件，用 $\Omega_i > \cdots$ 或 $\Omega_i < \cdots$ 表示。	4 points 4 分

Solution:

$$v_f = \frac{M_A+M_B}{M_i+M_A+M_B} v_{in}$$

$$\Omega_f = \frac{M_i}{M_i+M_A+M_B} \Omega_i$$

After the ejection of particle A, particle A has $v_{out} = (v_x, v_y)$, where $v_x = v_f$ and $v_y = \Omega_f R$.

Thus, the requirement $K_{out} > K_{in}$ can be expressed as

$$\Omega_i > \frac{v_{in}}{M_i R} \sqrt{\frac{(M_A+M_B)(M_i^2+M_B^2+2M_iM_A+2M_iM_B+M_AM_B)}{M_A}}$$

Suppose a civilization uses a rotating black hole as a power plant, by repeatedly using the model and process described in D2, and make use of the difference in incoming and outgoing kinetic energy.

假设一个文明利用旋转黑洞来提取能量。提取能量的方法是不断使用 D2 中描述的过程，以便利用出射和入射物体的动能差。

The black hole is initially at rest with mass M and angular velocity Ω. Each time, the civilization throws composite particle AB with the same initial velocity v_{in} with respect to the civilization themselves. The composite particle AB has mass $M_A, M_B \ll M$. Again, to simplify the calculation, just as in D2, we model the black hole by a uniform ring with fixed radius R although its mass grows by absorbing B.

初始时刻，黑洞静止，质量为 M，角动量为 Ω。每一次，这个文明把一个复合粒子 AB 用相同的初速度 v_{in} (相对于这个文明自己的参考系) 扔到黑洞中。设复合粒子的质量 $M_A, M_B \ll M$。为简化计算，正如 D2 中一样，我们把黑洞简化为匀质、半径固定为 R 的圆环。由于吸收 B 粒子，圆环质量增加。

This process is repeated as long as energy can be extracted from the black hole. When the process is repeated, the civilization keeps at rest in the v_{in} direction (the horizontal direction in the figure in D2), but follows the motion of the black hole in the directions perpendicular to v_{in}.

我们重复这个过程，直到不再能从黑洞中提取能量为止。当重复这个过程的时候，这个文明在 v_{in} 方向保持静止 (即 D2 题图中的水平方向)，但在垂直于 v_{in} 的方向上跟随黑洞运动。

D3	At the moment when no net energy can be extracted from the black hole, the civilization stops throwing matter in. What is the terminal angular velocity Ω_T of the black hole when the civilization stops throwing matter in? 当不再能用这个过程从黑洞中提取能量时，这个文明停止将物体抛入黑洞。当这个文明不再向黑洞抛射物体后，求黑洞末状态的角速度 Ω_T。	6 points 6 分

Solution:

Denote the total matter A, B the civilization has thrown up to a time be M_Σ, and $M_B = \gamma(M_A + M_B)$. At this time, the black hole mass is $M + \gamma M_\Sigma$.

Thus, for the matter that the civilization throws each time, $M_A = (1-\gamma)dM_\Sigma$, $M_B = \gamma dM_\Sigma$.

The differential momentum conservation equation in the horizontal direction is

$$v_{BH} + dv_{BH} = \frac{dM_\Sigma}{M+\gamma M_\Sigma} v_{in} + \frac{M+\gamma M_\Sigma + dM_\Sigma}{M+\gamma M_\Sigma} v_{BH}.$$

Thus, $v_{BH} = (1-x)\, v_{in}$, where $x \equiv \left(\frac{M}{M+\gamma M_\Sigma}\right)^{\frac{1}{\gamma}}$.

The differential angular momentum conservation equation is

$$(M + \gamma M_\Sigma + dM_\Sigma)(\Omega_{BH} + d\Omega_{BH}) = (M + \gamma M_\Sigma)\Omega_{BH}.$$

Thus, $\Omega_{BH} = x\,\Omega$.

At the time the civilization decides to stop throwing matter in,

$$0 = dE = \frac{1}{2}\frac{M_A}{M}(1-x)^2 v_{in}^2 dM_\Sigma + \frac{1}{2}\frac{M_A}{M} R^2\Omega^2 x^2 dM_\Sigma - \frac{1}{2} v_{in}^2 dM_\Sigma$$

Solve this equation, only taking the $x > 0$ solution x_+, we get

$$x_+ = \frac{1 + \sqrt{1 + \left(1 + \frac{R^2\Omega^2}{v_{in}^2}\right)\frac{M_B}{M_A}}}{1 + \frac{R^2\Omega^2}{v_{in}^2}}, \text{ and } \Omega_T = x_+\Omega.$$

Note:

1. The argument of black hole entropy here is a bit hand-waving. One should use quantum theory of vacuum fluctuation around black hole geometry in general relativity to actually calculate Hawking radiation and black hole entropy.

2. Throughout the solutions, we have used $\alpha \sim 1$ to estimate order of magnitudes. In fact, $\alpha = 4\pi^2$. Thus, the order of magnitudes estimation is off by this amount. If you are interested, you can easily insert $\alpha = 4\pi^2$ to get more precise estimations in the above questions.