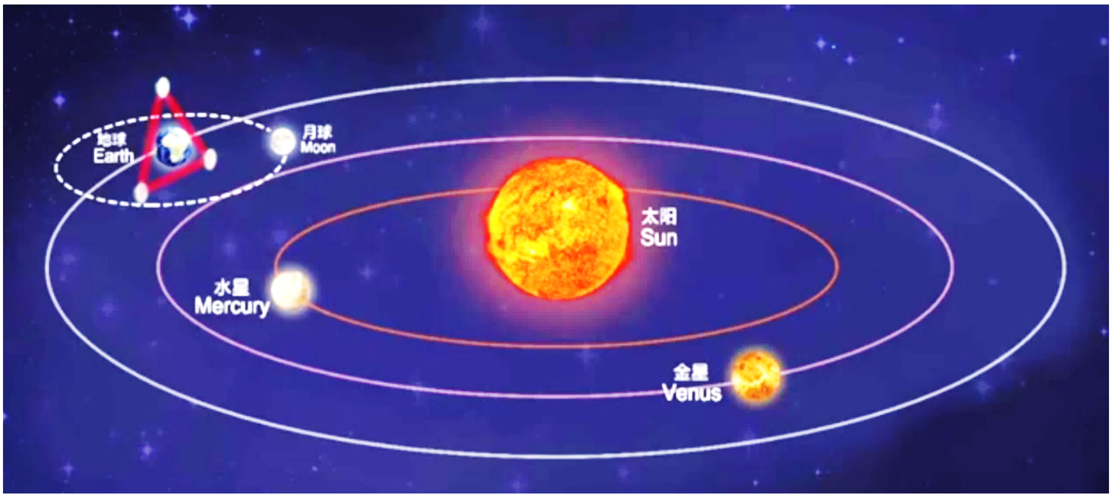
Problem 1: Error estimates of a gravitational wave experiment (30 points)

问题 1: 引力波测量中的误差估计 (30 分)

The discovery of gravitational waves initiated an era of gravitational wave astronomy. In addition to the ground-based gravitational wave observatories, gravitational wave observatories based on laser interference between satellites are also planned, for example, the Taiji and Tianqin programs in China and LISA in Europe. Here, we study a simplified version similar to the Tianqin program.
引力波的发现，开启了引力波天文学时代。除了在地面上建设引力波天文台，目前，通过卫星之间激光干涉的空间引力波计划也在筹划之中，例如中国的太极计划、天琴计划，和欧洲的 LISA。这里，我们考虑类似天琴引力波探测计划的一个简化版本。



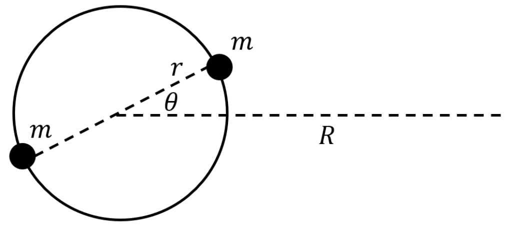
As illustrated in this figure, we consider three satellites surrounding the earth following circular orbits. They form an equilateral triangle. They form an interferometry in the nearly vacuum environment near the earth. From the change of interference patterns, the change of space distance is measured to detect gravitational waves. Here we will study the error sources for Tianqin to reach its desired measurement precision.
我们考虑如图所示，环绕地球呈等边三角形的三颗卫星按圆轨道运动，在地球周围接近真空的环境中组成激光干涉仪。通过激光的干涉条纹变化，来感知时空距离随时间的变化，探测引力波。本题将讨论，为了达到引力波探测精度，需要考虑的误差来源。

In this problem, we will use the physical constants and satellite parameters including:
在本题中将用到的物理参数和卫星技术参数包括：

Newton's gravitational constant 牛顿万有引力常数 $G = 6.67 \times 10^{-11} \mathrm{m}^3/(\mathrm{kg\ s}^2)$
Planck's constant 普朗克常数 $h = 6.626 \times 10^{-34} \mathrm{m}^2\mathrm{kg/s}$
Vacuum Permeability 真空磁导率 $\mu_0 = 1.257 \times 10^{-6}\ \mathrm{kg\ m\ s^{-2}\ A^{-2}}$
The mass of the earth 地球质量 $M = 5.97 \times 10^{24}\ \mathrm{kg}$
The radius of the earth 地球半径 $r = 6.37 \times 10^6 \mathrm{m}$
The distance from a satellite to the center of the earth 卫星轨道与地心的距离 $R = 10^8\ \mathrm{m}$
The laser wavelength used by the satellite 卫星使用激光波长 $\lambda = 1064\ \mathrm{nm}$
The size of the optical system of the satellite 卫星光学系统尺度 $D = 0.1\ \mathrm{m}$

Part A: Gravitational fluctuations on the orbit of the satellite 卫星轨道上的引力扰动

A1	Here we only consider gravity from the earth and consider the earth as a homogeneous ideal ball. Give the periodicity T of a satellite rotating around the earth. Please use second as the unit and give three significant figures. 在仅考虑地球引力，且设地球是均匀理想球体的情况下，求卫星绕地球转动的周期 T，请给出以	2 Points 2 分

	秒为单位的具体数值，精确到三位有效数字。	
A2	Since the shape and density of the earth is inhomogeneous, the satellite will feel an additional acceleration δa in addition to the uniform circular motion. To simplify the calculation, let us model the inhomogeneity of the earth as follows: Consider an ideal ball with mass $M-2m$. Two additional point masses (each has mass m) are put to diametrically opposite points on the equator of the earth. Assume that the satellite orbit and the earth are in the same plane, with angle θ between them. Give the precise formula to calculate δa. A2. 由于地球形状与密度的不均匀性，会对卫星产生除匀速圆周运动之外的额外加速度δa。为简化计算，将地球的不均匀性建模为：质量为 $M-2m$ 的理想球体，其赤道对径点上放置两个额外的质点，每个质点质量为 m。且假设卫星的轨道与地球赤道在同一平面内，与两质点夹角为θ。由此给出 δa 的精确计算公式。  	2 Points 2 分
A3	Calculate all the possible periodicities for δa. Please use second as the unit and give three significant figures. A3. 求 δa 随时间变化所有可能周期的数值，请给出以秒为单位的具体数值，精确到三位有效数字。	2 Points 2 分
A4	In the $R \gg r$ limit, give the leading order expression (the lowest nonzero order in the Taylor expansion of r/R) for δa. 求在 $R \gg r$ 极限下，δa 的领头阶（对r/R进行泰勒展开的最低的非零阶）表达式。	2 Points 2 分
A5	Estimate the typical value of δa (an error within two orders of magnitude will be considered as correct). 估计 δa 的典型数值（误差在两个量级之内可视为正确）。	3 Points 3 分
A6	In satellite experiments, we are interested in the gravitational waves with a particular periodicity (such as periods between 1-1000 seconds). Thus, if the periodicity of the gravitational fluctuation is too long, it will not interfere the gravitational wave measurement. Assume the satellite is co-rotating in the same direction with the spinning direction of the earth. In the Taylor expansion of δa, calculate the component with period closest to 1000s. Denote this component as δa_{1000}. Estimate the value of δa_{1000} / δa for $\theta = \pi/3$. (an error within two orders of magnitude will be considered as correct) A6. 在卫星实验上，我们感兴趣特定变化周期（例如周期为 1-1000 秒）的引力波信号。所以，变化周期太慢的引力扰动并不对引力波测量造成干扰。设卫星与地球自转方向相同，求 δa 的泰勒展开中周期最接近 1000s 的分量 δa_{1000} 与 δa 的比例 δa_{1000} / δa 。请给出该比例 δa_{1000} / δa 当 $\theta = \pi/3$ 时的数值，误差在两个量级内可视为正确。	3 Points 3 分

A1 Solution:

$\frac{v^2}{R} = \frac{GM}{R^2}, v = \sqrt{\frac{GM}{R}},$

$T = \frac{2\pi R}{v} = 3.15 \times 10^5$s.

A2. Solution：

$$\delta a_x = -\frac{G(M-2m)}{R^2} - \frac{Gm\,(R - r\cos\theta)}{(R^2+r^2-2\,R\,r\cos\theta)^{\frac{3}{2}}} - \frac{Gm\,(R + r\cos\theta)}{(R^2+r^2+2\,R\,r\cos\theta)^{\frac{3}{2}}}$$

$$\delta a_y = \frac{Gm\,r\sin\theta}{(R^2+r^2-2\,R\,r\cos\theta)^{\frac{3}{2}}} - \frac{Gm\,r\sin\theta}{(R^2+r^2+2\,R\,r\cos\theta)^{\frac{3}{2}}}$$

A3. Solution:

Note that the above expression has a symmetry of $\cos\theta \leftrightarrow -\cos\theta$. Thus in θ, the period is π (i.e., 1/2 of the period of the 2π rotation). In time, there are two possibilities: the earth co-rotate or counter-rotate with the satellite. Note that the earth

period is a day = $86400s$. Thus, the two possible periods are
$\frac{1/2}{\frac{1}{86400}\pm\frac{1}{3.147\times10^5}} = 3.39\times10^4\text{ s (counterrotate)}, or\ 5.95\times10^4\text{ s (corotate)}.$

A4: Solution:

Taylor-expand to second order (note that the first order in r/R result cancels):

$\delta a_x \simeq \frac{3Gmr^2(1-3\cos^2\theta)}{R^4}.$

$\delta a_y \simeq \frac{3Gmr^2\sin 2\theta}{R^4}.$

A5: Solution:

There can be many reasonable ways to estimate. For example, one can consider the density difference between the rock and the sea. Thus m can be estimated using

10^3kg m^{-3} × (average depth of the sea $3.5\times10^3 m$) × (10% of the earth surface area) $\sim 1.8\times10^{20}$. Thus, for a typical value of θ, for which $2-8\cos^2\theta$ doesn't cancel to extremely small values,

$\delta a\ \sim 10^{-9}\text{ m/s}^2$ (Note: the same order of magnitude can be obtained using precise modeling of the earth.
Answers ranging from $10^{-7}\sim10^{-11}$ can be considered correct.

A6: Solution:

Considering that in the co-rotating case, the period of the system is 119098s. To extract the component with 1000s period, Taylor-expand to the $(\cos\theta)^{120}$ term (note: there is no $(\cos\theta)^{119}$ term. Expanding to $(\cos\theta)^{118}$ is equally fine).
Then for n=120:

Note that the $\frac{3}{2}\left(\frac{3}{2}+1\right)\cdots\left(\frac{3}{2}+(n-1)\right)$ almost cancels the 1/n! in the Taylor expansion (the difference is at the order $\sqrt{n}$, which is within the error allowance).

$$\frac{\delta a_{1000}}{\delta a} = \frac{\frac{2^{n+1}\ R^n r^n\cos^n\theta}{R^{2n+2}}}{\delta a} = 10^{-141}$$

Note: taking n=118 the result is 10^{-139}, also considered correct. Also note that in reality (when we go beyond the current toy model), the higher order multiples will dominate δa_{1000}.

Part B: Free electrons from the solar wind 太阳风中的自由电子

Consider the laser signal between the satellites. Although the space between satellites is close to the vacuum, but it is not the absolute vacuum. In particular, solar wind will introduce free electrons. Let the number density of the free electrons be N_e, the electric charge of an electron be e，the electron mass be m_e. And we ignore other media apart from these electrons.

考虑卫星之间的激光信号。虽然卫星之间的环境真空度较高，但并不是绝对的真空。特别地，太阳风会带来自由电

子。设自由电子的数密度为 N_e，电子电量为e，电子质量为 m_e，且忽略卫星之间除自由电子以外其它的介质。

B1	Assume that the electrons move freely in the electric field produced by the laser. Calculate the acceleration of the electron $d\mathbf{v}_e/dt$ as a function of the electric field $\mathbf{E}$ produced by the laser. 设电子在激光产生的电场中自由运动，求电子的加速度 $d\mathbf{v}_e/dt$ 与激光产生的电场 $\mathbf{E}$ 之间的关系。	1 Points 1 分
B2	Calculate the time dependence of the current from the free electrons. 求自由电子带来的电流随时间的变化 $d\mathbf{J}/dt$。	2 Points 2 分
B3	Calculate the phase speed of the laser v_p in the environment of the free electrons (since v_p is very close to the speed of light, the higher order difference between v_p and the speed of light can be ignored). Hint: from the Maxwell equations, one can derive that $\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0.$ B3. 求激光在自由电子中运动的相速度 v_p（由于 v_p 足够接近光速， v_p 与光速差别的高阶项可以忽略）。 提示：由麦克斯韦方程组，可以推出 $\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0.$	3 Points 3 分
B4	Let $N_e = 10\ \mathrm{cm}^{-3}$. Calculate the phase error of the laser between two satellites. In other words, if there were no free electrons, the laser waveform arrived at a satellite is $\cos\theta$. Now with free electrons, the same wave form at the same moment changes into $\cos(\theta+\delta\theta)$. Calculate the value of $\delta\theta$. 设 $N_e = 10\ \mathrm{cm}^{-3}$，求两个卫星之间，由自由电子引起的激光相位误差。即假设没有自由电子，到达卫星波形为 $\cos\theta$ ，有自由电子的情况下，在同一时刻到达卫星波形为 $\cos(\theta+\delta\theta)$ ，求 $\delta\theta$ 的数值。	3 Points 3 分

B1: Solution:

$$\frac{d\mathbf{v}_e}{dt} = -e\frac{\mathbf{E}}{m_e}$$

B2: Solution:

$$\frac{d\mathbf{J}}{dt} = -e\,N_e\frac{d\mathbf{v}_e}{dt} = \frac{N_e e^2\mathbf{E}}{m_e}$$

B3: Solution:

From $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \nabla\times\mathbf{B} = \mu_0\,\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$,

$\frac{\partial^2\mathbf{E}}{\partial t^2} - c^2\nabla^2\mathbf{E} + c^2\,\mu_0\frac{d\mathbf{J}}{dt} = 0$ (where $\nabla\times(\nabla\times\mathbf{E}) = \nabla^2\mathbf{E} - \nabla(\nabla\cdot\boldsymbol{E}) \simeq \nabla^2\mathbf{E}$)

It's solution in frequency space is

$\omega^2 = c^2k^2 + c^2\mu_0\frac{N_e e^2}{m_e}$.

Thus, the phase velocity

$v_p = \frac{\omega}{k} = \sqrt{c^2 + c^2\mu_0\frac{N_e e^2}{m_e k^2}} \simeq \sqrt{c^2 + c^2\mu_0\frac{N_e e^2\lambda^2}{4\pi^2\,m_e}} \simeq c\left(1 + \frac{\mu_0 N_e e^2\lambda^2}{8\pi^2\,m_e}\right)$

B4: Solution:

The wave can be written as $\cos(kx - \omega t + \theta)$, where θ is the initial phase at the emitter $t = 0, x = 0$. The receiver has a distance $x = \sqrt{3}\,R$ and the arrival time is $t = \sqrt{3}\,R/c$.

With free electrons, $k = \frac{\omega}{c}\left(1 - \frac{\mu_0 N_e e^2\lambda^2}{8\pi^2\,m_e}\right)$. Thus inserting x and t,

$\delta\theta = -\frac{\omega}{c} \times \frac{\mu_0 N_e e^2 \lambda^2}{8\pi^2 m_e} \times \sqrt{3}\, R = -\frac{\mu_0 N_e e^2 \lambda}{4\pi\, m_e} \times \sqrt{3}\, R = 5.19 \times 10^{-6}$

Part C: Shot noise 散粒噪声

Any precision measurements are limited by the uncertainty principle of quantum mechanics. Assume that every photon's arrival time at the detector can be considered as independent stochastic processes. Also, in actual experiments, phase error of the laser is more important. But here for simplicity, here we only estimate photon number errors.

再精确的测量手段，都要受到量子力学的制约。设卫星干涉仪中，激光中每个光子到达探测器的时间都是独立的随机事件。另外，实验中其实更关心激光的相位误差，但是这里我们为简便起见，仅估计光子数误差。

C1	During a certain period of time, the average photon number in the laser is N. In this case, the error in the photon number measurement in the laser is $\Delta N = N^{\alpha}$. Find α. C1. 某段时间内，激光中平均包含 N 个光子。此时对激光中光子数的测量的误差为 $\Delta N = N^{\alpha}$。求 α。	1 Points 1 分
C2	If we request that in one second, the relative error of photon number measurement is $\frac{\Delta N}{N} < 3 \times 10^{-6}$. Calculate the minimal power of laser P_{rec} that the satellite should receive. (3 points) C2. 若要求在一秒钟时间内，光子数测量的相对误差为 $\frac{\Delta N}{N} < 3 \times 10^{-6}$，求卫星接收到的激光最低功率 P_{rec}。	3 Points 3 分
C3	Assume the laser arrived at a satellite is emitted from the other satellite from the three-satellite system. Estimate: in an ideal case, what is the minimal emission power of laser P_{emit} from the other satellite (can be considered to be correct if the order-of-magnitude is correct). 设到达卫星的激光是由三卫星系统中，另一个卫星上的激光器发射的。估计理想状况下，激光的最低发射功率 P_{emit}（量级正确即可视为正确）。	3 Points 3 分

C1: Solution：$\alpha = 1/2$.

C2: Solution:

We thus need the minimal number of photon $N = 1.11 \times 10^{11}$ per second.

The energy of each photon is $E = h\,\nu = h\frac{c}{\lambda} = 1.87 \times 10^{-19} J$

Thus the minimal power at reception is $P_{\text{rec}} = 2 \times 10^{-8}\text{W}$.

C3: Solution 1: Using Gaussian laser beam and the formula

$\frac{P_{\text{rec}}}{P_{\text{emit}}} = \left(\frac{\pi\, D^2}{4\, R\, \lambda}\right)^2 = 5.45 \times 10^{-9}$.

Thus, $P_{\text{emit}} \sim 3.67\text{W}$

Solution 2: Estimation from the uncertainty principle

Within the satellite optics system diameter D, for each photon, the momentum uncertainty of the laser is $\Delta p \sim h/D$. Thus the minimal spread angle is λ/D. Over a distance R, the minimal radius of the spot is $\lambda\, R/D$. Considering the receiver radius is at most D, too. Thus,

$\frac{P_{\text{rec}}}{P_{\text{emit}}} \sim \left(\frac{D}{R\frac{\lambda}{D}}\right)^2 \sim 8.83 \times 10^{-9}$.

Thus, $P_{\text{emit}} \sim 2.26\text{W}$

Problem 2: Metric-modified geodesic and heat conduction (30 points)
问题 2: 度量修正的测地线和热传导

Solving physics, such as wave propagation, geodesics, and thermal conduction, on a curved surface in 3D requires a thorough understanding of metrics and differential geometry. However, there can be significant simplifications for systems with spatial symmetry or by adopting coordinate transformation. In this question, we will go through two problems for physics on a curved surface. The first one is light propagating on a curved surface. Figure 1 (a) shows a circular cone with height $5\ \mathrm{mm}$ and a base diameter of $2\rho_0 = 10$ mm, joining to a flat surface. The flat surface has a circular hole of the same diameter so that as a whole, there is only one single surface with the cone part indicating the 'curved space.' The entire surface, including the flat surface and the cone, have a very thin surface so that light can be effectively confined on such a surface. We have assumed the cone is joint smoothly to the flat surface. The second question, being illustrated later, is about steady-state thermal conduction on a hemispherical surface.

在 3D 曲面上解决物理问题，如波传播、测地线和热传导，需要对度量和微分几何有深刻的理解。然而，对于具有空间对称性的系统或采用坐标变换的情况，可以进行显著的简化。在这个问题中，我们将讨论在曲面上解决的两个物理问题。第一个问题是光在曲面上传播。图 1（a）显示了一个高度为 5 mm、底直径为$2\rho_0 = 10$ mm 的圆锥体与一个平面相连接。平面上有一个相同直径的圆孔，以便整体上只有一个单一的表面，圆锥部分表示“曲面”。整个表面，包括平面和圆锥体，都有非常薄的表面，以便光可以有效地限制在这样的表面上。我们假设锥体与平坦表面平稳连接。第二个问题将在后面进行阐述，它涉及到在半球面上的稳态热传导。

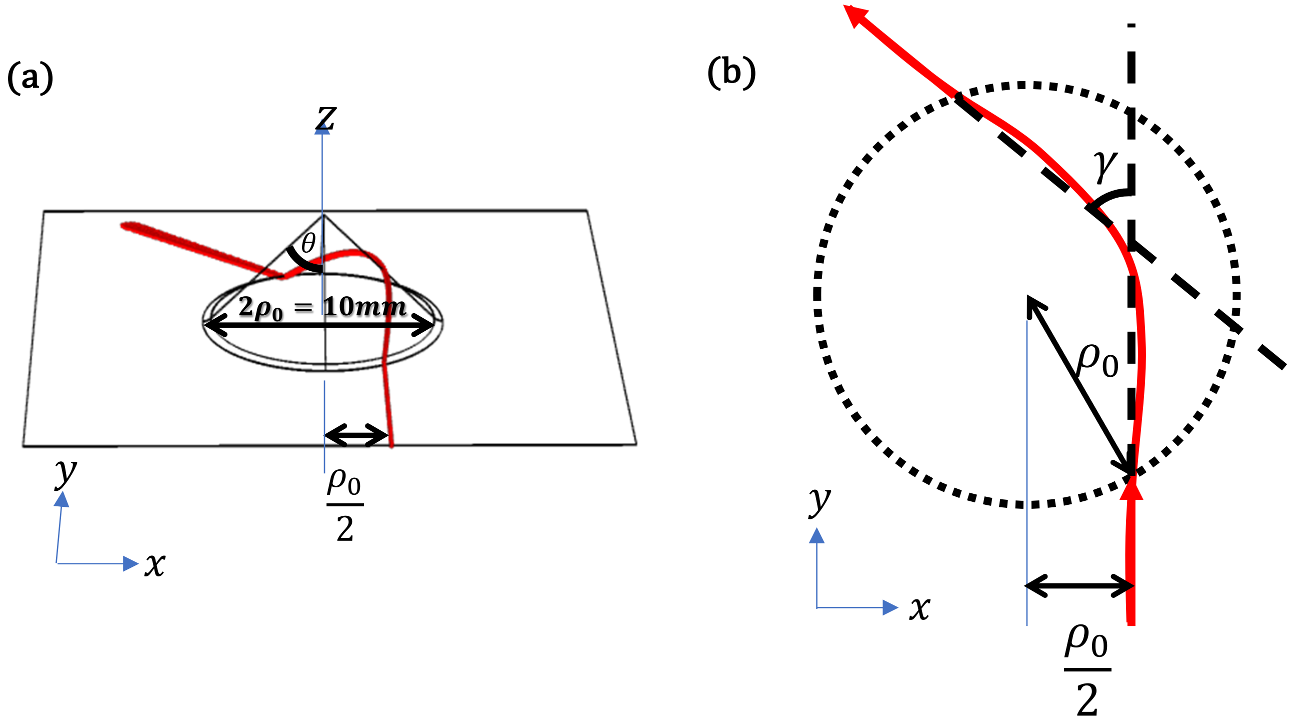


Figure 1(a) depicts a curved surface created by connecting a circular cone to a flat surface with a hole of the same size as the cone's base. In Figure 1(b), we present a top view of this surface. Light confined to such a surface originates on the flat surface at the bottom, undergoes bending due to the cone, and exits in a different direction.

图 1(a)展示了由连接到一个底部与圆锥体底部相同大小的圆孔的平面上的圆锥体所定义的曲面。图 1(b)是该曲面的俯视图。被限制在这样的曲面上的光从底部的平面上开始，因圆锥体的弯曲而改变方向，最终以不同的方向出射。

A. GEODESIC ON A ROTATIONAL SYMMETRIC CURVED SURFACE 旋转对称曲面上的测地线

In mechanics, we are aware that when a system exhibits rotational symmetry, we can simplify the derivation of dynamics by applying the conservation of angular momentum. For instance, we can employ the conservation of angular momentum to derive Kepler's laws. In the current scenario, we consider a normalized angular momentum, denoted as L, which is defined as:

在力学中，我们知道当系统具有旋转对称性时，我们可以使用角动量守恒来简化问题。例如，我们可以使用角动量守恒来推导开普勒定律。在当前情况下，我们考虑一个归一化的角动量，记为 L，其定义如下：

$$L = \hat{z} \cdot \boldsymbol{\rho} \times \frac{d\boldsymbol{\rho}}{ds} = \rho^2 \frac{d\phi}{ds}.$$

Here, $\boldsymbol{\rho}$ represents the projected position vector on the two-dimensional x-y plane, given by $\boldsymbol{\rho} = x\hat{x} + y\hat{y} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$, with the projected cone center as the origin. ρ is the magnitude of vector $\boldsymbol{\rho}$ and s is the arc length along the path of light on the surface.
在这里，$\boldsymbol{\rho}$ 代表了在二维 x-y 平面上的投影位置矢量，由$\boldsymbol{\rho} = x\hat{x} + y\hat{y} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$给出，其中投影锥体中心为原点。$\rho$是矢量$\boldsymbol{\rho}$ 的大小。s 是沿着曲面上光的路径的弧长。

A1	Given that the infinitesimal arc length on the cone satisfies $ds^2 = dx^2 + dy^2 + dz^2$ and $z = z(\rho)$ is the height at that point, prove the geodesic on the cone satisfies 假设圆锥体上的无穷小弧长满足$ds^2 = dx^2 + dy^2 + dz^2$，其中$z = z(\rho)$是该点的高度，请证明圆锥体上的测地线满足 $$\rho'(\phi)^2 = \frac{\sin^2\theta}{L^2}\rho^2(\rho^2 - L^2)$$ Instead of using arc length s to parametrize the geodesic, we have used ϕ for parametrization. 这里，我们使用ϕ来参数化测地线，而不是使用弧长 s。	3 points
A2	For a light starting on the flat surface with a perpendicular distance of $\rho_0/2$ to the origin, what will be minimal ρ the light can go. 对于从平面上距离原点垂直距离为 $\rho_0/2$ 的入射光线，它能够到达的最小 ρ 是多少？	2 points
A3	What is the deflection angle γ by comparing the entering and exit rays on the flat surface? Hint: you may need to use $\int \frac{dx}{x\sqrt{x^2-1}} = \tan^{-1}\sqrt{x^2-1} + c$ 通过比较光线在平面表面的入射和出射角度，计算偏转角度γ。 提示：你可能需要用$\int \frac{dx}{x\sqrt{x^2-1}} = \tan^{-1}\sqrt{x^2-1} + c$	7 points

A1: Coordinates:

$$x = \rho\cos\phi\,, \qquad y = \rho\sin\phi\,, \qquad z = z(\rho)$$
$$\Rightarrow \frac{dx}{ds} = \frac{d\rho}{ds}\cos\phi - \rho\sin\phi\frac{d\phi}{ds}, \qquad \frac{dy}{ds} = \frac{d\rho}{ds}\sin\phi + \rho\cos\phi\frac{d\phi}{ds}$$

Along the light path, the arc length elapsed in a small section is governed by

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= (\cos\phi\, d\rho - \rho\sin\phi\, d\phi)^2 + (\sin\phi\ d\rho + \rho\cos\phi\, d\phi)^2 + z'(\rho)^2(d\rho)^2 \\ &= (1 + z'(\rho)^2)(d\rho)^2 + \rho^2(d\phi)^2 \end{aligned}$$

Angular momentum:

$$L = \hat{z}\cdot\boldsymbol{\rho}\times\frac{d\boldsymbol{\rho}}{ds} = \rho\cos\phi\frac{dy}{ds} - \rho\sin\phi\frac{dx}{ds} = \rho^2\frac{d\phi}{ds}$$
$$\Rightarrow \frac{\rho^4}{L^2} = \left(\frac{ds}{d\phi}\right)^2 = (1 + \cot^2\theta)\rho'(\phi)^2 + \rho^2 = \csc^2\theta\,\rho'(\phi)^2 + \rho^2$$
$$\Rightarrow \rho'(\phi)^2 = \sin^2\theta\,\rho^2\left(\frac{\rho^2}{L^2} - 1\right)$$

A2: The trajectory on the flat surface can be described by

$$x = \frac{\rho_0}{2}, \qquad y = s + s_0,$$

where s_0 is a constant. Then the normalized angular momentum is

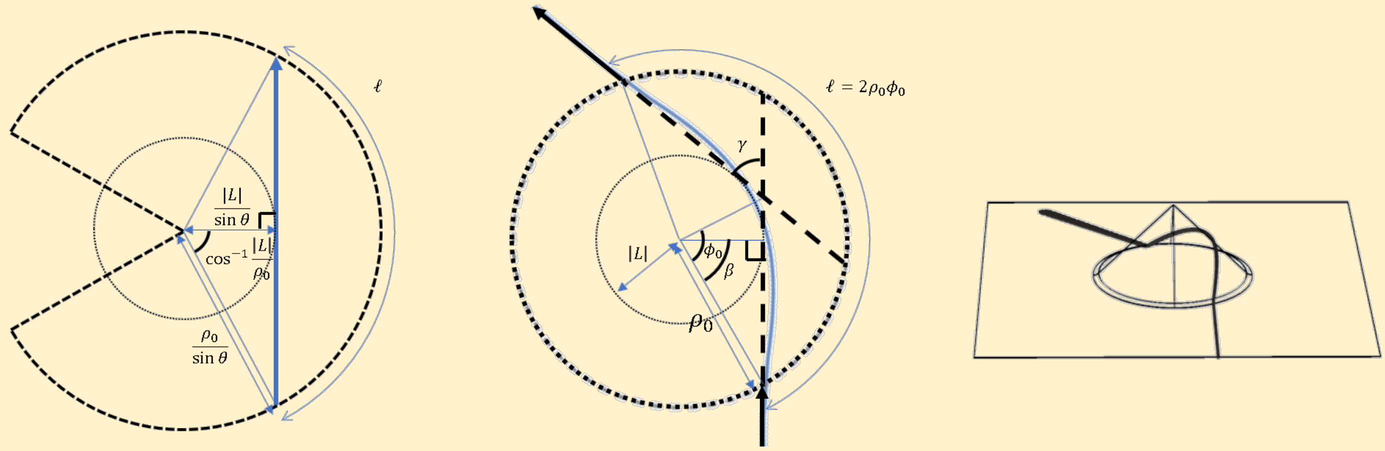
$$L = x\frac{dy}{ds} - y\frac{dx}{ds} = \frac{\rho_0}{2} \ ,$$

which is independent of s as expected on the flat surface before entering the cone. We recognize that L has also the meaning of perpendicular distance of the entering ray. Then from A1,

$$\rho'(\phi)^2 \geq 0 \Rightarrow \rho \geq |L| = \frac{\rho_0}{2}$$

A3: Solution: We consider that L is a constant of motion so that the entering and exit rays have the same perpendicular distance $|L|$. We define $2\phi_0$ as the angle elapsed for the ray in the region of the cone and $2\beta = 2\cos^{-1}(|L|/\rho_0)$ as the angle elapsed for the ray if it has not been deflected (see middle panel).

Flattened out **Projection of 3D on x-y plane** **3D**



(Method 1)

To calculate ϕ_0, we flatten out the cone to a flat surface. The cone becomes a sector of radius $\rho_0/\sin\theta$. The nearest projected distance $|L|$ (in A2) now becomes $|L|/\sin\theta$ (see left panel). Therefore, the elapsed angle subtends a distance on the edge of cone as

$$\ell = \frac{2\rho_0}{\sin\theta}\cos^{-1}\frac{|L|}{\rho_0}$$

Back to the projected view (middle panel), we have the same distance $\ell = 2\rho_0\phi_0$ so that we have obtained

$$\phi_0 = \frac{1}{\sin\theta}\cos^{-1}\frac{|L|}{\rho_0} = \frac{\beta}{\sin\theta}$$

(Method 2) Alternatively, we can consider the ray has its ρ gradually decreasing when it enters the cone until the nearest distance, $|L|$ from A2, for the first half of the ray trajectory, we choose the negative square root from A1:

$$\frac{d\rho}{d\phi} = -\sin\theta\,\rho\sqrt{(\rho/|L|)^2 - 1}$$

Now, we can integrate to get ϕ_0

$$\phi_0 = \int d\phi = -\frac{1}{\sin\theta}\int \frac{d\rho/|L|}{(\rho/|L|)\sqrt{(\rho/|L|)^2 - 1}} = \left[-\frac{\tan^{-1}\sqrt{\rho^2/L^2 - 1}}{\sin\theta}\right]_{\rho=\rho_0}^{\rho=|L|}$$

$$= \frac{\tan^{-1}\sqrt{\rho_0^2/L^2 - 1}}{\sin\theta} = \frac{1}{\sin\theta}\cos^{-1}\frac{|L|}{\rho_0}$$

After calculating ϕ_0:

Then the deflection angle γ is governed by

$$\gamma = 2(\phi_0 - \beta) = 2\left(\frac{1}{\sin\theta} - 1\right)\beta.$$

Now, substituting values for our particular example, $|L| = \rho_0/2$ and $\theta = \pi/4$, we have

$$\beta = \cos^{-1}\frac{|L|}{\rho_0} = \frac{\pi}{3} = 60°.$$

$$\gamma = \left(\sqrt{2} - 1\right)\frac{2\pi}{3} = 49.7°.$$

as the deflection angle.

Reference: The experiment and the flattened-out model are depicted in Phys. Rev. Appl. 11, 034072 (2019).

B. HEAT CONDUCTION ON A SPHERICAL SURFACE (I) 球面上的热传导（I）

A usual trick is to search for a coordinate transform from the curved surface (represented by the Cartesian coordinates (x, y, z)) to a 2-dimensional coordinates system $X - Y$ plane so that the physics on the (X, Y) just looks like a flat plane. For a unit spherical surface, such a map is the stereographic projection

一個有用的技巧是寻找一个坐标变换，将曲面（由笛卡尔坐标(x, y, z) 表示）映射到一个二维的 $X - Y$ 坐标上，使得在 $X - Y$ 坐标上的物理现象看起来就像一个平面。对于一个单位球面，这样的映射是立体投影。

$$(X, Y) = \left(\frac{x}{z+1}, \frac{y}{z+1}\right) = (\rho\cos\phi, \rho\sin\phi)$$

Suppose now we consider heat conduction problem on such a spherical surface, i.e. a very thin shell of spherical surface. The steady-state heat conduction has the temperature profile satisfying the Laplace equation

现在我们考虑在这样一个球面上的热传导问题，即一个非常薄的球面壳体。稳态热传导具有满足拉普拉斯方程(Laplace equation)的温度分布。

$$\nabla^2 T(\theta, \phi) = 0$$

while temperature profile is independent of radial distance r in spherical coordinate (r, θ, ϕ). The spherical surface is at radius $r = 1$.

其中温度分布与径向距离 r 无关。这个球面的半径为 $r = 1$。

B1	Prove T satisfies Laplace equation on the (X, Y) coordinate: 证明 T 在 (X, Y) 坐标上满足 Laplace 方程: $$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho T) + \frac{1}{\rho^2}\partial_\phi^2 T = 0$$ Hint: For convenience, we are given the Laplacian in spherical and cylindrical coordinates as 为了方便起见，我们提供了球坐标和柱坐标下的 Laplacian 方程式。 In spherical coordinate (在球坐标下) (r, θ, ϕ): $$\nabla^2 f = \frac{1}{r^2}\,\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f$$ cylindrical coordinate (在柱坐标下) (ρ, ϕ, z): $$\nabla^2 f = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho f) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$$	4 points

Solution:

For the stereographic projection, a θ is mapped to a ρ and ϕ is unaltered, serving both the azimuthal coordinate for both the spherical coordinate and the stereographic projected coordinate. Then,

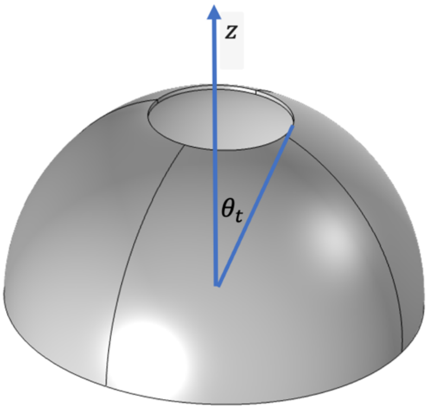
$$\rho = \frac{\sin\theta}{\cos\theta + 1} = \tan\frac{\theta}{2}$$
$$\Rightarrow \frac{\partial\rho}{\partial\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}, \qquad \sin\theta\frac{\partial\rho}{\partial\theta} = \rho$$
$$\Rightarrow \sin\theta\,\partial_\theta = \rho\partial_\rho$$

For a r-independent T profile, it satisfies the Laplace equation at $r = 1$ as

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta T) + \frac{1}{\sin^2\theta}\partial_\phi^2 T = 0 \Rightarrow \sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta T) + \partial_\phi^2 T = 0.$$

which is now transformed to

$$\rho\partial_\rho(\rho\partial_\rho T) + \partial_\phi^2 T = 0. \quad \Rightarrow \frac{1}{\rho}\partial_\rho(\rho\partial_\rho T) + \frac{1}{\rho^2}\partial_\phi^2 T = 0$$



Now, the above figure gives the thin shell in the shape of lamp shade, which is in a hemi-spherical surface with a circular opening at the top. The whole shape still has a rotational symmetry about the vertical z-axis. The bottom of the lamp shade is kept at temperature T_b (at $\theta = \frac{\pi}{2}$ for spherical polar coordinate) and the top is kept at temperature T_t (at $\theta = \theta_t$).

现在，上图给出了一个薄壳，呈灯罩形状，是一个顶部有一个圆形开口的半球面。整个形状仍然具有旋转对称性。灯罩的底部保持在温度 T_b（在球坐标下的 $\theta = \frac{\pi}{2}$ 处），顶部保持在温度 T_t（在$\theta = \theta_t$ 处）。

B2	Solve the temperature profile, as a function of θ, with such rotational symmetry. 解出具有旋转对称性的温度分布 $T(\theta)$，作为 θ 的函数。	4 points

Solution:

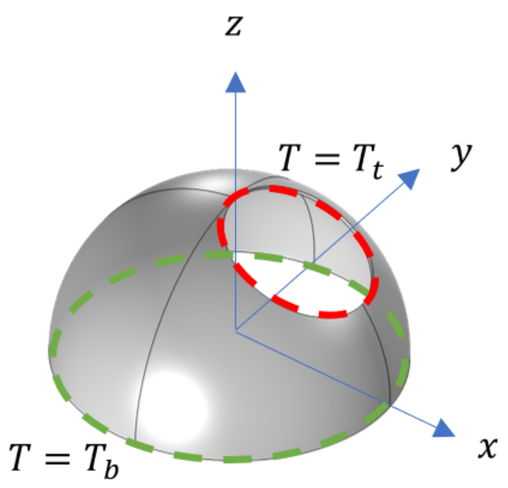
With rotational symmetric, T is independent of ϕ:

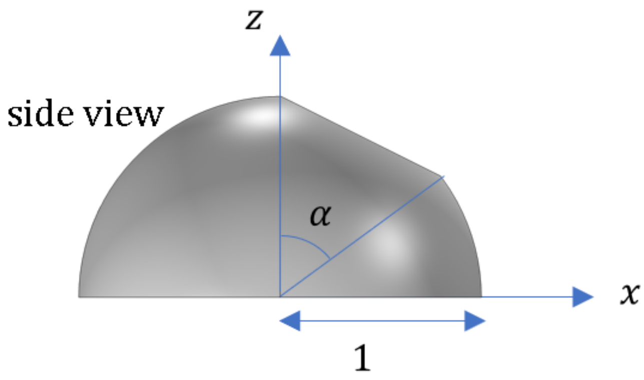
$$\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho T\right) = 0.$$

Then

$$T = \alpha \ln \rho + \beta = (T_t - T_b)\frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\theta_t}{2}\right)} + T_b$$

C. HEAT CONDUCTION ON A SPHERICAL SURFACE (II) 球面上的热方程(II)





Now, we consider the top opening is tilted about the y-axis in breaking rotational symmetry. Suppose the top opening is still a circle on the spherical surface passing through $(x, y, z) = (0,0,1)$ and $(\sin\alpha, 0, \cos\alpha)$ as diameter and its normal is on the x-z plane.
现在，我们考虑顶部开口围绕 y 轴倾斜，破坏了旋转对称性。假设球面顶部开口仍然是一个圆，通过由 $(x, y, z) = (0,0,1)$ 和 $(\sin\alpha, 0, \cos\alpha)$ 作为直径，并且其法向量位于 x-z 平面上。

C1	Determine where the top opening is mapped on (X, Y) plane through the stereographic projection map. 通过立体投影映射，确定顶部开口在 (X, Y) 平面上的映射位置。 Hint: the answer is still a circle in X and Y coordinates. 提示：答案在 X 和 Y 坐标上仍然是一个圆。	2 points
C2	Solve the temperature profile $T(X, Y)$ when the bottom opening is kept at temperature T_b and the top is kept at temperature T_t. You can leave your answer in terms of X and Y coordinates. Hint: In the stereographic projected domain X-Y plane, Laplace equation is satisfied and you can further use general method of image, like the one used in solving electrostatic problem by putting two point charges on the X-axis with undetermined charges. 解出温度分布函数 $T(X, Y)$。底部开口保持温度T_b，顶部开口保持温度T_t。您可以用 X 和 Y 坐标表示您的答案。 提示：Laplace 方程仍然满足在立体投影后的 X-Y 平面上。你可以使用在解决静电问题時的镜像电荷法，在 X 轴上放置两个待确定电荷的电荷。	8 points 8 分

C1: The two points of a diameter:

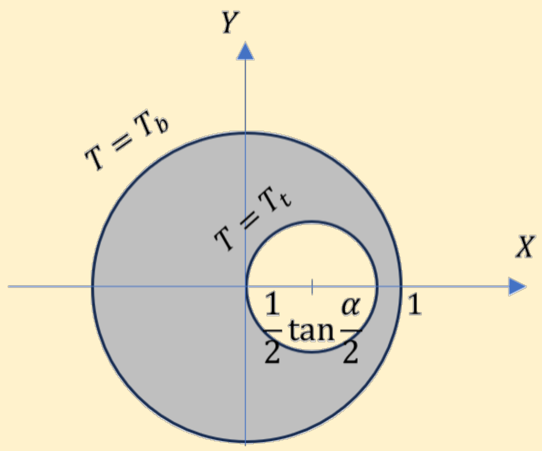
$$(x, y, z) = (0,0,1) \rightarrow (X, Y) = (0,0)$$

$$(x, y, z) = (\sin\alpha, 0, \cos\alpha) \rightarrow (X, Y) = \left(\tan\frac{\alpha}{2}, 0\right)$$

Mapped circle has center $(X, Y) = \left(\frac{1}{2}\tan\frac{\alpha}{2}, 0\right)$ with diameter $\frac{1}{2}\tan\frac{\alpha}{2}$, or written as

$$\left(X - \frac{1}{2}\tan\frac{\alpha}{2}\right)^2 + Y^2 = \frac{1}{4}\tan^2\frac{\alpha}{2}$$

C2:



The Laplace equation is satisfied on the X-Y plane. $T = T_b$ on the unit circle with center at origin. $T = T_t$ on a smaller circle passing through origin and with center at $\left(a = \frac{1}{2}\tan\frac{\alpha}{2}, 0\right)$.

Equivalently to our thermal conduction problem, we can treat T as an electrostatic potential also satisfying the Laplace equation. Here, we adopt the general method of image to obtain T in (X, Y) space. Assume that we have two point charges on the X-axis with undetermined charges at the moment. We let the distance between them is $2A$ and the middle of the two coordinates is at X_0 on the X-axis. Then, the potential for such a system is

$$T = c_1 \ln\left(\frac{(X - X_0 + A)^2 + Y^2}{(X - X_0 - A)^2 + Y^2}\right) + c_2.$$

An equipotential (same value of T) contour can be written as a circle with center on the X-axis:

$$\frac{(X - X_0 + A)^2 + Y^2}{(Y - X_0 - A)^2 + Y^2} = K \Rightarrow \left(X - \left(X_0 + \frac{K+1}{K-1}A\right)\right)^2 + Y^2 = \left(\left(\frac{K+1}{K-1}\right)^2 - 1\right)A^2$$

where K is an arbitrary number depending on the value of T. Now, we need the inner and outer circles in (X, Y) space map to different equipotential lines. Then, we have following equations for the centres and radii,

$$X_0 + \frac{K_1+1}{K_1-1}A = a, \quad \left(\frac{K_1+1}{K_1-1}\right)^2 - 1 = \frac{a^2}{A^2},$$
$$X_0 + \frac{K_2+1}{K_2-1}A = 0, \quad \left(\frac{K_2+1}{K_2-1}\right)^2 - 1 = \frac{1}{A^2},$$

Eliminating K_1 and K_2, we have

$$(a - X_0)^2 = A^2 + a^2, \qquad X_0^2 = A^2 + 1,$$

in which X_0 and A can be solved as

$$X_0 = \frac{1}{2a}, \qquad A = \pm\frac{\sqrt{1 - 4a^2}}{2a}$$

We let

$$\mu = \frac{1 + \sqrt{1 - 4a^2}}{2a} > 1, \qquad \mu^{-1} = \frac{1 - \sqrt{1 - 4a^2}}{2a} < 1$$

Then

$$T = c_1 \ln\left(\frac{(X-\mu)^2 + Y^2}{(X-\mu^{-1})^2 + Y^2}\right) + c_2$$

We probe the c_1 and c_2 coefficients by

$$(X,Y) = (0,0) \Rightarrow T_t = 4c_1 \ln \mu + c_2$$
$$(X,Y) = (1,0) \Rightarrow T_b = 2c_1 \ln \mu + c_2$$
$$\Rightarrow c_1 = \frac{T_t - T_b}{2\ln \mu}, c_2 = 2T_b - T_t$$

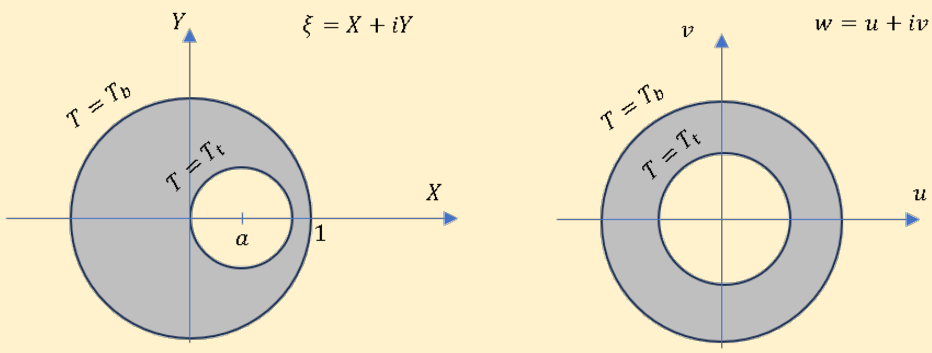
Therefore

$$\begin{aligned} T &= \frac{T_t - T_b}{2\ln\mu} \ln\left(\frac{(X-\mu)^2 + Y^2}{(X-\mu^{-1})^2 + Y^2}\right) + 2T_b - T_t \\ &= \frac{T_t - T_b}{2\ln\mu} \ln\left(\frac{\rho^2 - 2\mu\rho\cos\phi + \mu^2}{\rho^2 - 2\mu^{-1}\rho\cos\phi + \mu^{-2}}\right) + 2T_b - T_t \\ &= \frac{T_t - T_b}{2\ln\mu} \ln\left(\frac{\tan^2\frac{\theta}{2} - 2\mu\ \tan\frac{\theta}{2}\cos\phi + \mu^2}{\tan^2\frac{\theta}{2} - 2\mu^{-1}\tan\frac{\theta}{2}\cos\phi + \mu^{-2}}\right) + 2T_b - T_t \end{aligned}$$

where $\mu = \frac{1+\sqrt{1-4a^2}}{2a}$, $a = \frac{1}{2}\tan\frac{\alpha}{2}$.

Alternative solution:

If you know conformal map which maps circle to circle and preserves Laplace equation, we can adopt a conformal map (coordinate transformation) from (X,Y) to (u,v) so that the two circles are concentric.



By writing complex $\xi = X + iY$ and $w = u + iv$, we seek

$$w = \frac{b\xi + c}{\xi + d}$$

which maps the unit circle to unit circle and the inner circle to another circle with common center at $w = 0$

$$\frac{b+c}{1+d} = 1, \qquad \frac{-b+c}{-1+d} = -1$$
$$\frac{0+c}{0+d} + \frac{2ab+c}{2a+d} = 0$$

By solving b, c and d from the above, we obtain the conformal map as

$$w = \frac{\mu\xi - 1}{\mu - \xi}$$

or

$$w = \frac{\mu - \xi}{\mu\xi - 1}$$

We choose the first solution which maps the inner circle to a circle with radius less than one, just for convenience. The radius of the inner circle is now mapped to a circle with radius $|(0-1)/(\mu-0)| = \mu^{-1} < 1$.

From previous experience in B2 in solving ϕ-independent Laplace equation, we have

$$\begin{aligned}
T &= (T_t - T_b)\frac{\ln|w|}{\ln \mu^{-1}} + T_b \\
&= \frac{T_t - T_b}{2\ln \mu^{-1}} \ln\left|\frac{\mu\xi - 1}{\mu - \xi}\right|^2 + T_b \\
&= \frac{T_t - T_b}{2\ln \mu^{-1}} \ln\left|\frac{\xi - \mu^{-1}}{\mu - \xi}\right|^2 + 2T_b - T_t \\
&= \frac{T_t - T_b}{2\ln \mu} \ln\left|\frac{\xi - \mu}{\xi - \mu^{-1}}\right|^2 + 2T_b - T_t \\
&= \frac{T_t - T_b}{2\ln \mu} \ln\left(\frac{(X-\mu)^2 + Y^2}{(X-\mu^{-1})^2 + Y^2}\right) + 2T_b - T_t
\end{aligned}$$