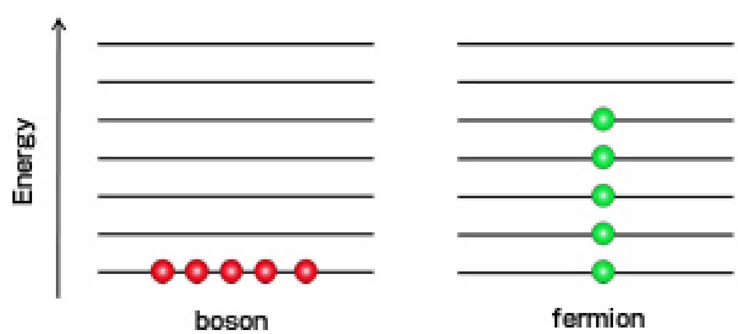
Pan Pearl River Delta Physics Olympiad 2017
2017 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-2 (Total 2 Problems, 55 Points) 简体版卷-2（共2题，55分）
(2:00 pm – 5:00 pm, 3 February, 2017)

Problem 1: Bose Einstein Condensation (22 points) 玻色-爱因斯坦凝聚（22 分）

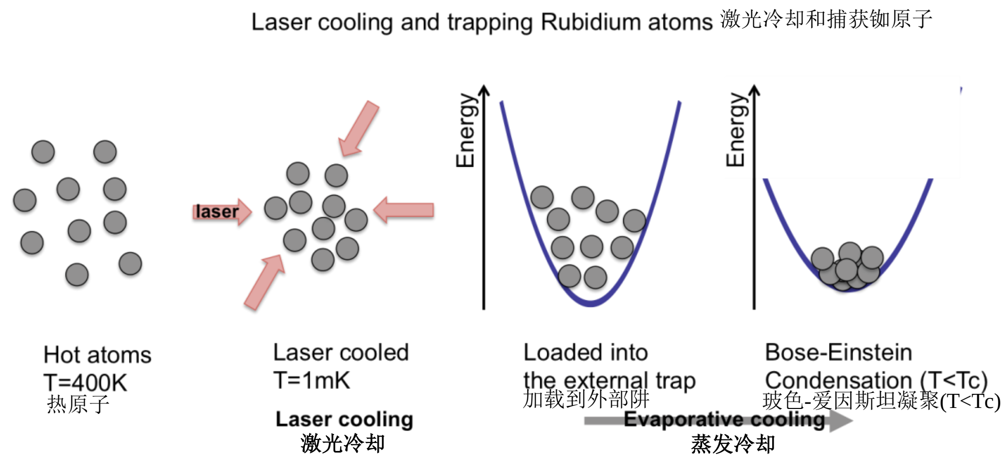
Planck's constant 普朗克常数 $h = 6.626 \times 10^{-34}$ Js
Boltzmann constant 波尔兹曼常数 $k_B = 1.381 \times 10^{-23}$ JK^{-1}

In nature, particles are classified into two different kinds: bosons and fermions. Bosons (e.g. photons) are particles that like to be together in the same state. In contrast, fermions (e.g. electrons, protons and neutrons) are unlikely to go into an already occupied state according to the Pauli exclusion principle. Statistical mechanics tells us that when a system of bosons reaches a critical density in a trap it undergoes a transition that a large number of bosons will have a tendency to occupy the same lowest-energy state. This phenomenon is called Bose-Einstein condensation. The following figure shows how bosons and fermions occupy energy states when the temperature approaches 0 K.
在自然界中，粒子可以分为两种不同的类型：玻色子和费米子。玻色子（例如光子）是喜欢一起处于相同状态的粒子。相反，根据泡利不相容原理，费米子（例如电子、质子和中子）不可能进入已经被占据的状态。统计力学告诉我们，当一个玻色子系统在阱中达到临界密度时，它会经历相变，令大量的玻色子倾向占据相同的最低能阶。这种现象称为玻色-爱因斯坦凝聚。下图显示当温度接近 0 K 时，玻色子和费米子如何占据能阶。



Recent development of trapping and cooling ultracold atoms (e.g. Sodium, Rubidium and Lithium atoms) paved the way for the observation of Bose-Einstein condensation of atomic gases in ultracold temperature (Nobel prize in physics 2001), which had been theoretically predicted by Bose and Einstein in 1924. Several different cooling techniques have been employed to achieve ultracold temperature around 10-100 nK (note $1\ \text{nK} = 10^{-9}\text{K}$). For example, the hot Rubidium atoms prepared at 400 K are cooled down to ~1mK through the Laser cooling techniques (Nobel prize in physics in 1997). Such cold atoms prepared by laser cooling technique are typically loaded into the external trap (produced by either magnetic or optical fields) for further cooling as shown below.

在捕获和冷却超冷原子（例如钠、铷和锂原子）的技术上，近年的进展为观察超冷温度下原子气体的玻色-爱因斯坦凝聚（2001 年诺贝尔物理学奖）提供了有利条件，印证了 1924 年玻色和爱因斯坦的预测。几种不同的冷却技术已被采用以实现约 10-100 nK 的超冷温度（注意 1 nK = 10^{-9}K）。例如，通过激光冷却技术（1997 年诺贝尔物理学奖），在 400K 下制备的热铷原子可以冷却至～1 mK。这种冷原子通常被加载到外部阱（由磁场或光场产生）中，用於进一步冷却，如下所示。



A. Maxwell- Boltzmann distribution and the thermal de Broglie wavelength of the atoms 麦克斯韦-玻尔兹曼分布和原子的热德布罗意波长

Consider a dilute gas of atoms. The inter-particle interactions are very weak. In this case, the gas can be described by the ideal gas model in which the particles move freely inside a stationary trap without interacting with one another except for very brief elastic collisions to reach thermal equilibrium.
考虑稀释的原子气体。粒子间相互作用非常弱。在这种情况下，气体可以通过理想气体模型描述，其中粒子在固定阱内自由移动，除了在趋向热平衡的过程中会有非常短暂的弹性碰撞，彼此没有相互作用。

In this atomic gas system, the probability distribution of the particle speed v is given by Maxwell-Boltzmann distribution,
在这种原子气体系统中，粒子速度v 的概率分布由麦克斯韦 - 玻尔兹曼分布给出，

$$f(v) = \sqrt{\left(\frac{m}{2\pi k_B T}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2k_B T}},$$

where m is the mass of the atom, k_B is the Boltzmann constant, and T is the temperature of the gas in the unit of Kelvin [K].
其中 m 是原子的质量，k_B是玻尔兹曼常数，T是气体温度，单位为[K]。

A1	Derive the most probable velocity v_{mp} of a particle at temperature T.	**2 points**

	试推导温度为 T 时粒子最可能的速度v_{mp}。	**2 分**

$f'(v) = 0 \Rightarrow v_{mp} = \sqrt{\frac{2k_B T}{m}}$

(Remarks: 1 point if only $f'(v) = 0$ is given)

A2	Based on the most probable velocity v_{mp} obtained in A1, write down the characteristic de Broglie wavelength λ_{dB} of the particle in an atomic gas at temperature T. 根据在 A1 中求得的最可能速度v_{mp}，试写下温度为 T 时原子气体中粒子的特征德布罗意波长λ_{dB}。	**2 points 2 分**

The de Broglie wavelength of the particle is

$$\lambda_{\mathrm{dB}} = \frac{h}{mv} \text{ (1 point)}$$

The characteristic de Broglie wavelength is estimated by replacing the velocity by the characteristic velocity v_{mp} as

$$\lambda_{\mathrm{dB}} = \sqrt{\frac{h^2}{2mk_B T}} \text{ (1 point)}$$

Since particles in a gas of atoms have different speed following Maxwell-Boltzmann distribution, it is useful to consider the <u>*thermal de Broglie wavelength* (λ_T)</u> defined as $\lambda_T = \lambda_{dB} \times \pi^{-\frac{1}{2}}$. Here we derive the Bose-Einstein temperature T_C for a gas of N non-interacting (bosonic) atoms of mass m in a three-dimensional box with volume V. We will consider the simple physical picture that Bose-Einstein condensation occurs when the characteristic inter-particle distance between bosonic atoms becomes comparable to the thermal de Broglie wavelength λ_T. (Planck's constant h = 6.626 × 10^{-34} Js, Boltzmann constant k_B = 1.381 × 10^{-23} JK^{-1})
由于原子气体中的粒子按著麦克斯韦-波尔兹曼分布，各有不同的速率，我们引入<u>热德布罗意波长(λ_T)</u>，定义为$\lambda_T = \lambda_{dB} \times \pi^{-\frac{1}{2}}$。在这里，我们会考虑在体积为 V 的三维盒子中的原子气体，其中有 N 个质量为 m 的非相互作用（玻色子）原子，我们会推导其玻色-爱因斯坦温度 T_C 。我们将採用一幅简单的物理图画，就是当玻色子原子间的特征距离和热德布罗意波长λ_T相若时，玻色-爱因斯坦凝聚便会发生。

A3	What is the expected T_C of the $N = 10^5$ atoms of mass $m = 1.445 \times 10^{-25}$ kg trapped in the trap with a volume of $V = 10^5 \mu\mathrm{m}^3$? (1 μm^3 = 10^{-18} m^3) 在体积为 $V = 10^5 \mu\mathrm{m}^3$的阱中，捕获 $N = 10^5$ 个质量为 $m = 1.445 \times$	**3 points 3 分**

	10^{-25} kg的原子，求 T_C 的预期值。(1 μm^3 = 10^{-18} m^3)	

At the Bose-Einstein temperature T_C, the inter-particle separation is equal to the de Broglie wavelength as

$\lambda_{dB} = \sqrt{\frac{h^2}{2\pi m k_B T}} \approx \left(\frac{V}{N}\right)^{\frac{1}{3}}$ (1 point for identifying the expression of the inter-particle separation)

Therefore, the T_C is given by

$$T_c = \frac{h^2}{2\pi m k_B}\left(\frac{N}{V}\right)^{\frac{2}{3}} \quad \text{(1 point)}$$

(Remarks: Note that this is an estimate, hence any final numerical result from an estimation containing $\sim \frac{h^2}{k_B m}\left(\frac{N}{V}\right)^{\frac{2}{3}}$ can be regarded as correct.)

For the given parameters, $N = 10^5$ atoms and trap volume $V = 10^5$ μm^3, $T_c \sim 35$ nK. (1 point)

(Note: The exact result is $T_c \approx 0.527 \frac{h^2}{2\pi m k_B}\left(\frac{N}{V}\right)^{\frac{2}{3}}$.

B. Evaporative cooling in an external trap 在外部阱中的蒸发冷却

The temperatures reached by laser cooling are extremely low (< 1 mK), but they are not cold enough to realize Bose-Einstein condensation. To date, Bose-Einstein condensation of alkali atoms has been achieved by using evaporative cooling after atoms are loaded into the external trap. During evaporative cooling, when atoms escaping from a trap have a kinetic energy higher than the average energy of atoms in the trap, the remaining atoms become cooled.

激光冷却达到的温度极低（< 1 mK），但还是不够冷去实现玻色-爱因斯坦凝聚。到目前为止，碱金属原子的玻色-爱因斯坦凝聚可以通过把原子加载到外部阱之后，使用蒸发冷却实现。在蒸发冷却期间，当从阱中逸出的原子具有高于阱中原子平均能量的动能时，剩余的原子就会冷却。

In the following problems in part B, we will estimate the effect of evaporative cooling. For atoms trapped in a box of fixed volume and having no heat exchange with the surroundings, we assume that an average energy of trapped atoms is ϵ and a small number of atoms $|\Delta N|$ are evaporated within a short time $\Delta\tau$ with an average energy of $(1+\beta)\epsilon$ where $\beta > 0$. During the process, the small change in the number of atoms $\Delta N < 0$ leads to the change $\Delta\epsilon < 0$ in the average energy of the remaining atoms. We also assume that $\left|\frac{\Delta\epsilon}{\epsilon}\right| \ll 1$ and $\left|\frac{\Delta N}{N}\right| \ll 1$.

在下面 B 部的问题中，我们将估计蒸发冷却的影响。对于被捕获在固定体积的盒子中，并且没有与周围环境进行热交换的原子，我们假设被捕获原子的平均能量是ϵ。假设有小数目的原子$|\Delta N|$ 短时间$\Delta\tau$内蒸发，其平均能量为$(1+\beta)\epsilon$，其中$\beta > 0$。在这过程中，原子数目$\Delta N < 0$ 的小变化，导致剩余原子的平均能量的变化$\Delta\epsilon < 0$。我们还假设$\left|\frac{\Delta\epsilon}{\epsilon}\right| \ll 1$和$\left|\frac{\Delta N}{N}\right| \ll 1$。

[Remark: In the derived relation, you may ignore the term $\frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N}$ since $\frac{\Delta\epsilon}{\epsilon} \ll 1$ and $\left|\frac{\Delta N}{N}\right| \ll 1$.]

［备注：在推导的关系中，由于$\frac{\Delta\epsilon}{\epsilon} \ll 1$和$\left|\frac{\Delta N}{N}\right| \ll 1$，可以忽略 $\frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N}$一项 。］

B1	Derive the relation between $\Delta\epsilon$ and ΔN with β, ϵ and N. 试用 β、 ϵ和 N，推导 $\Delta\epsilon$和 ΔN之间的关系。	**3 points** **3 分**

The average energy per atom after atom loss ΔN must be $\epsilon + \Delta\epsilon$.
Total (mechanical) energy of $(N + \Delta N)$ atoms after atom loss = $(N\epsilon + (1+\beta)\epsilon\Delta N)$
Therefore, we have a relation:

$$\epsilon + \Delta\epsilon = \frac{N\epsilon + (1+\beta)\epsilon\Delta N}{N + \Delta N} \qquad \text{(1 point)}$$

Then

$$\beta\frac{\Delta N}{N} = \frac{\Delta\epsilon}{\epsilon} + \frac{\Delta\epsilon}{\epsilon}\frac{\Delta N}{N} \qquad \text{(1 point for steps)}$$

and

$$\beta\frac{\Delta N}{N} = \frac{\Delta\epsilon}{\epsilon} \qquad \text{(1 point)}$$

by ignoring the second order term as $\left|\frac{\Delta\epsilon}{\epsilon}\right|\left|\frac{\Delta N}{N}\right| \ll 1$.

Now we consider cold atoms at the initial temperature of $T_i = 200\mu\text{K}$ in a trap. Assume that we remove 1% of atoms (i.e. $\left|\frac{\Delta N}{N}\right| = 0.01$) during each time period $\Delta\tau$ and $\beta = 2$.
现在我们考虑阱中的冷原子，初始温度为 $T_i = 200\mu\text{K}$。假设我们在每段时间 $\Delta\tau$期间去除1%的原子（即$\left|\frac{\Delta N}{N}\right| = 0.01$），并且$\beta = 2$。

B2	Then estimate the final temperature T_f of atoms after the evaporative cooling over the total time period of $350\Delta\tau$. 试估计在$350\Delta\tau$的总时间段内，经蒸发冷却后的原子的最终温度T_f 。	**3 points** **3 分**

From the result of B1, we know $\beta\frac{\Delta N}{N} = \frac{\Delta T}{T}$. (1 point)
For each time period $\Delta\tau$, the temperature changes:

$$T_{\text{after}} = T_{\text{before}} + \Delta T = T_{\text{before}}\left(1 + \frac{\Delta T}{T_{\text{before}}}\right) = T_{\text{before}}\left(1 + \frac{\beta\Delta N}{N}\right) \qquad \text{(1 point)}$$

Here note that $\Delta T < 0$ and $\Delta N < 0$.

Therefore, after 350 time period $\Delta\tau$,

$$T_{\text{final}} = \left(1 - \frac{\beta\Delta N}{N}\right)^{350} T_{\text{initial}} = (1 - 2\times 0.01)^{350} \times (2\times 10^5)\text{nK} \qquad \text{(1 point)}$$

$$= 169.8 \text{ nK} \qquad \text{(1 point)}$$

[Remarks : Alternative approximation may give slightly different temperature. The final temperature T_{final} between 160 nK and 180 nK can be regarded as correct.]

C. Bose-Einstein temperature T_C in a harmonic potential
谐波势中的玻色-爱因斯坦凝聚温度 T_C

In a real experiment with ultracold atomic gases, a gas of bosonic atoms is trapped in a three-dimensional harmonic trap generated by the laser beam or the magnetic field. Here we consider a three-dimensional trap characterized by the harmonic potential:
在超冷原子气体的真实实验中，玻色原子气体被捕获在由激光束或磁场产生的三维谐波阱中。这里我们考虑一个三维阱，可用谐波势描述：

$$U_{\mathrm{trap}} = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

C1	Consider the fact that ultracold atoms are oscillating around the bottom of the trap with the characteristic trapping frequency $\omega_i/2\pi$ along the i-direction. Derive the characteristic volume confining the atoms in terms of T and $\omega_{x,y,z}$. 考虑超冷原子在阱底振荡，沿着 i 方向的特征捕获频率为 $\omega_i/2\pi$ 。试推导原子被限定的特征体积，答案以 T 和$\omega_{x,y,z}$表达。	**3 points** **3 分**

Along the x-axis, the characteristic kinetic energy of the atom is given by $\frac{1}{2}k_BT$ from the kinetic theory of the ideal gas. Then the maximum displacement R_X of the atom trapped in the harmonic potential $U_{\mathrm{trap}}^x = \frac{1}{2}m\omega_x^2R_x^2$ is given by

$$\frac{1}{2}k_BT = \frac{1}{2}m\,\omega_x^2R_x^2 \quad \text{(2 points)}$$

and

$$R_x = \sqrt{\frac{k_BT}{m\omega_x^2}}$$

In a similar way, one can derive

$$R_{y,z} = \sqrt{\frac{k_BT}{m\omega_{y,z}^2}}$$

Therefore, the characteristic volume V is given

$$V \sim R_xR_yR_z = \frac{\left(\frac{k_BT}{m}\right)^{\frac{3}{2}}}{\omega_x\omega_y\omega_z} \quad \text{(1 point)}$$

[Remarks: Note that this is an estimate, hence any final numerical result from an estimation containing $\sim \frac{\left(\frac{k_BT}{m}\right)^{\frac{3}{2}}}{\omega_x\omega_y\omega_z}$ can be regarded as correct.]

C2	Derive the Bose-Einstein condensation temperature T_C of the atoms trapped in a harmonic trap considered in Part C1 in terms of ω_i and N. 试推导 C1 部的谐波阱中捕获的原子的玻色-爱因斯坦凝聚温度 T_C，答案以 ω_i 和 N 表达。	**2 points** **2 分**

The characteristic inter-particle separation is given by

$$\left(\frac{V}{N}\right)^{\frac{1}{3}} \sim N^{-\frac{1}{3}}\left(\frac{k_B T}{m}\right)^{\frac{1}{2}}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}} \text{ (1 point)}$$

Considering $\lambda_T \sim \left(\frac{V}{N}\right)^{\frac{1}{3}} = \sqrt{\frac{h^2}{2\pi m k_B T}}$,

$$T_C = \frac{h}{k_B\sqrt{2\pi}}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}} N^{\frac{1}{3}} \text{ (1 point)}$$

Note: The exact result is $T_C \simeq 0.15\frac{h}{k_B}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}}N^{\frac{1}{3}}$.

[Remark: Note that this is an estimate, hence any final numerical result from an estimation containing $\sim\frac{1}{k_B}\,h\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}}N^{\frac{1}{3}}$ can be regarded as correct.]

C3	What is the Bose-Einstein condensation temperature T_c of the $N = 10^4$ atoms of mass $m = 1.445 \times 10^{-25}$ kg in the harmonic trap with trapping frequencies $\omega_x/2\pi = \omega_y/2\pi = \omega_z/2\pi = 100$ Hz? 谐波阱中有 $N = 10^4$个原子，每个原子的质量为$m = 1.445 \times 10^{-25}$ kg，谐波频率为 $\omega_x/2\pi = \omega_y/2\pi = \omega_z/2\pi = 100$ Hz。求玻色−爱因斯坦凝聚温度 T_c。	**1 point** **1 分**

From the result of part C2, $T_C = \frac{h}{k_B\sqrt{2\pi}}\left(\omega_x\omega_y\omega_z\right)^{\frac{1}{3}}N^{\frac{1}{3}}$.

Using $\omega_x = \omega_y = \omega_z = 2\pi \times 100$ rad/s, $N = 10^4$, $T_c = 259.1$ nK. (1 point)

Note that the evaporative cooling is efficient enough to achieve the Bose-Einstein condensation.
注意，蒸发冷却的效率足以实现玻色-爱因斯坦凝聚。

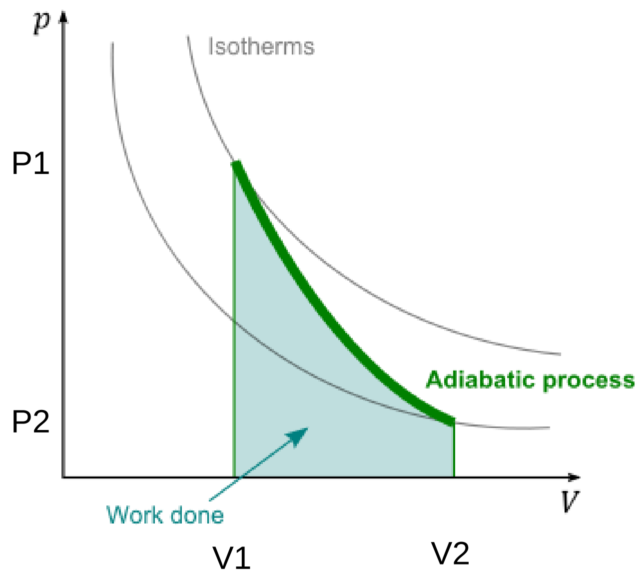
D. Adiabatic cooling by slowly expanding the trap 通过缓慢膨胀阱进行绝热冷却

Cooling atomic gases to lower temperature has been motivated by the quest to observe new forms of matter such as superfluid. However the evaporative cooling we discussed in part B is not always preferable since a number of atoms leave the trap during the process. In this part we consider a different cooling technique (so-called adiabatic cooling) by slowly expanding the trap without losing atoms.
把原子气体冷却至更低温度的动机，是寻求物质的新状态（例如超流体）。然而，我们在 B 部中讨论的蒸发冷却，不一定是首选的方法，因为在该过程中有许多原子离开了阱。在这部中，我们会考虑另一冷却技术（所谓的绝热冷却），是通过缓慢地膨胀阱而不损失原子而达成的。

D1	Calculate the fraction of atoms remaining in the trap after the evaporative cooling described in part B2. 试计算在 B2 部中描述的蒸发冷却之后，留在阱中的原子的分数。	**1 point** **1 分**

Fraction of remaining atoms $= (1-0.01)^{350} = 0.03$ (1 point)

Consider N atoms in an external harmonic trap with trapping frequencies of $\omega_x = \omega_y = \omega_z = 2\pi f_0$ at the temperature $T_1 = 105$ nK $= 1.05 \times 10^{-7}$ K. From now on, we assume that the whole atomic gas can be regarded as a monoatomic ideal gas. At this stage, the atomic gas has the pressure P_1 and the volume V_1 as described in the figure below.
考虑在外部谐波阱中有N个原子，捕获频率为$\omega_x = \omega_y = \omega_z = 2\pi f_0$，其温度为$T_1 = 105$ nK $= 1.05 \times 10^{-7}$ K。从现在开始，我们假设整个原子气体可视为单原子理想气体。这时，原子气体具有如下图所示的压强P_1和体积V_1。



Now consider the adiabatic decompression process of N atoms trapped in a harmonic trap. For this we adiabatically change the trapping frequencies of the harmonic potential trap from $\omega_{x,1} = \omega_{y,1} = \omega_{z,1} = 2\pi f_0$ to $\omega_{x,2} = 2\pi f_0$ and $\omega_{y,2} = \omega_{z,2} = \frac{2\pi f_0}{10}$ following the adiabatic process in the P-V diagram. Note that there is no heat exchange between the atomic gas and the environment (actually vacuum) and no atoms leave the trap during the process.
现在考虑捕获 N 个原子的谐波阱的绝热减压过程。为此，我们绝热地将谐波势阱的捕获频率从$\omega_{x,1} = \omega_{y,1} = \omega_{z,1} = 2\pi f_0$ 改变至 $\omega_{x,2} = 2\pi f_0$ 和 $\omega_{y,2} = \omega_{z,2} = \frac{2\pi f_0}{10}$。注意，原子气体和环境（实际上是真空）之间没有热交换，并且在该过程中没有原子离开阱。

D2	Calculate the final temperature of the atomic gas after adiabatic decompression of the trap. 试计算阱绝热减压后原子气体的最终温度。	**2 points** **2 分**

$PV^\gamma =$ constant and $PV = nRT$ implies $T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$. (1 point)
For the monoatomic gas with $\gamma = 5/3$. From the result of (C2), the volume changes as

$$V_2 = 100\, V_1$$

and thus

$$T_2 = \frac{T_1}{100^{\gamma-1}} = \frac{105}{100^{\frac{5}{3}-1}} = 5 \text{ nK (1 point)}$$

Remark: Some students first substituted the adiabatic relation into the result of part C2 to find the relation between the volume and the trapping frequencies. They found that V only expands 10 times and the final temperature is 23 nK. This answer has sound physics reasoning and is also considered correct. On the other hand, the question setter realized that in actual experiments, the volume does not change that dramatically on cooling due to the repulsive interaction. In retrospect, the problem might be less ambiguous by directly pointing out that the volume increases 100 times.

Remark: Using this adiabatic cooling via adiabatic decompression of the trap, researchers at MIT had achieved the coldest matter in universe around 500 pico-Kelvin in 2003 (research work reported in Science 301, 1513-1515 (2003)).

Acknowledgement: We thank Prof. Gyu-Boong Jo for contributing this interesting question.

Problem 2: Swimming Microorganisms (33 points) 游泳微生物（33 分）

Although objects in water tend to sink in a gravitational field, microorganisms such as paramecium can control their swimming directions not necessarily subject to gravitational field. Recently, physicists proposed that their swimming patterns are related to their asymmetric shape. When they swim in a viscous fluid, they experience asymmetric resistance forces that may cause them to rotate.
虽然水中的物体倾向于在重力场中下沉，但是诸如草履虫的微生物可以控制它们的游泳方向，不一定受到重力场的影响。最近，物理学家提出他们的游泳模式和他们不对称的形状有关。当它们在粘性流体中游泳时，它们经历可能导致它们旋转的不对称抗阻力。

A. Resistive Forces and Torques in a Viscous Fluid 粘性流体中的抗阻力和力矩

For a rod having a translational motion in a viscous fluid, there are two kinds of resistive forces. In this question, we will refer to the resistive force acting in the normal direction of the rod as the *drag*, and the resistive force along the direction of the rod as the *friction*, as shown in Fig. 1(a). The drag per unit length is approximated as $\mu v_\perp$, and the friction per unit length as $\frac{\mu v_\parallel}{2}$, where $v_\perp$ and $v_\parallel$ are the velocity components normal and parallel to the axis of the rod respectively, and μ is a constant proportional to the viscosity of the fluid.
在粘性流体中有平移运动的杆子，存在两种抗阻力。在本题中，我们将作用在杆子的法向方向上的抗阻力，称为阻力，而将作用在沿杆方向的阻力，称为摩擦力，如图 1(a)所示。每单位长度的阻力近似为$\mu v_\perp$，而每单位长度的摩擦力为$\frac{\mu v_\parallel}{2}$，其中 $v_\perp$ 和 $v_\parallel$分别为垂直和平行于杆轴线的速度分量，μ是与流体粘度成比例的常数。

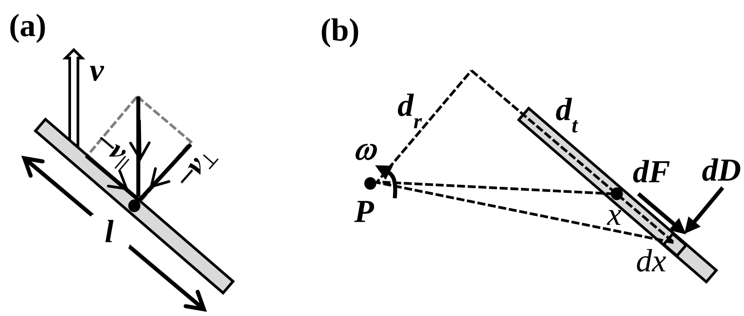


Fig. 1: (a) Directions of the resistive forces acting on a rod moving in a viscous fluid with velocity **v** indicated as the white arrow. The drag is directed along $-\mathbf{v}_\perp$, and the friction along $-\mathbf{v}_\parallel$. (b) The resistive forces acting on an element of the rod rotating about point P in the same plane at radial distance d_r and tangential distance d_t from its center.
图 1：（a）杆子在粘性流体中运动时，作用在杆子上的抗阻力的方向，速度 **v** 以白色箭头表示。阻力方向沿著$-\mathbf{v}_\perp$，摩擦力方向则沿著$-\mathbf{v}_\parallel$。（b）杆子围绕 P 点在同一平面旋转，杆子中心与 P 点的径向距离为 d_r，切向距离为 d_t。图示作用在杆子的一小段上的抗阻力。

As shown in Fig. 1(b), consider a reference point P whose radial and tangential distances from the center of the rod are d_r and d_t respectively. If the rod has a fixed position and orientation with respect to P, and P has a translational motion, then the resistive forces acting on the rod can be calculated using Fig. 1(a). However, if the rod also rotates in the same plane about P at an angular velocity ω, there will be extra forces and torques acting on the rod due to drag and friction.

如图 1（b）所示，考虑一个参考点 P，与杆子中心的径向和切向距离分别为 d_r 和 d_t。如果杆子相对于 P 点的位置和取向固定，并且 P 在作平移运动，则可使用图 1(a)计算作用在杆子上的抗阻力。然而，如果杆子也在同一平面以角速度ω围绕 P 转动，阻力和摩擦力将产生额外的力和力矩作用在杆子上。

A1	Derive the friction F due to the rotational motion. 试推导由于旋转运动引起的摩擦力 F。	**1 point** **1 分**

As shown in Fig. 1(c), $dF = \frac{\mu}{2}\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\frac{d_r}{\sqrt{(x+d_t)^2+d_r^2}} = \frac{1}{2}\mu d_r\omega dx$

$F = \frac{1}{2}\mu l d_r\omega$ [1]

A2	Derive the drag D due to the rotational motion. 试推导由于旋转运动引起的阻力 D。	**1 point** **1 分**

As shown in Fig. 1(c), $dD = \mu\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\frac{x+d_t}{\sqrt{(x+d_t)^2+d_r^2}} = \mu\omega(x+d_t)dx$

$D = \mu\omega\int_{-l/2}^{l/2}(x+d_t)dx = \mu l d_t\omega$ [1]

A3	Derive the torque τ_f about the axis of rotation due to the friction. 试推导摩擦力围绕旋转轴心的力矩 τ_f 。	**1 point** **1 分**

As shown in Fig. 1(c), $d\tau_f = \frac{\mu}{2}\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\left(\frac{d_r}{\sqrt{(x+d_t)^2+d_r^2}}\right)d_r = \frac{1}{2}\mu\omega d_r^2 dx$

Torque $\tau_f = \frac{1}{2}\mu l d_r^2\omega$ [1]

A4	Derive the torque τ_d about the axis of rotation due to the drag. 试推导阻力围绕旋转轴心的力矩 τ_d 。	**2 points** **2 分**

As shown in Fig. 1(c), $d\tau_d = \mu\left(\omega\sqrt{(x+d_t)^2+d_r^2}\,dx\right)\frac{x+d_t}{\sqrt{(x+d_t)^2+d_r^2}}(x+d_t) = \mu\omega(x+d_t)^2dx$

$\tau_d = \mu\omega\int_{-l/2}^{l/2}(x+d_t)^2dx = \mu\omega\int_{-l/2}^{l/2}(x^2+2d_tx+d_t^2)dx = \frac{1}{12}\mu l^3\omega + \mu l d_t^2\omega$ [1,1]

B. A Passive Microswimmer with No Rotation 无动力又不旋转的游泳微生物

An asymmetric microswimmer is L-shaped with the dimensions shown in Fig. 2(a). The mass of the microswimmer is m and the density is uniform. The lengths of the long and short arms are $4b$ and $2b$ respectively. The width and thickness of its two arms are negligible.
有不对称的游泳微生物具有 L 形的形状，尺寸如图 2（a）所示。游泳微生物的质量为 m，密度均匀。长臂和短臂的长度分别为 $4b$ 和 $2b$。两臂的宽度和厚度可忽略。

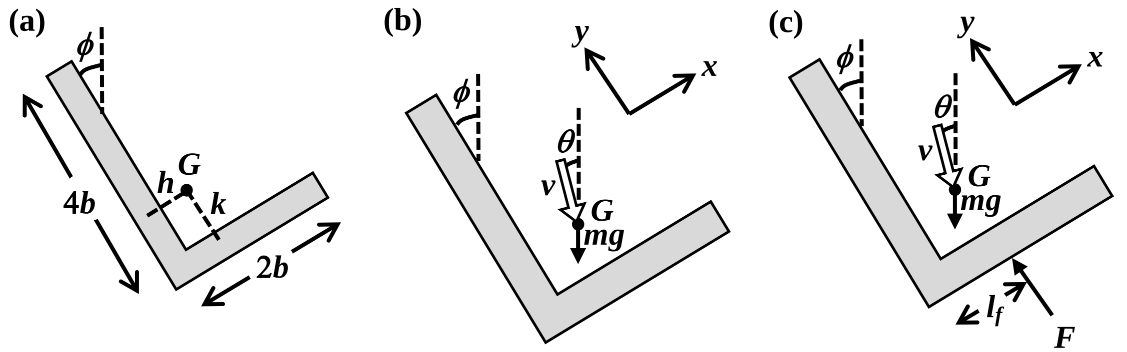


Fig. 2: (a) Dimensions of the microswimmer. (b) The weight and the velocity of a passive microswimmer. (c) An active microswimmer.
图 2:（a）游泳微生物的尺寸。（b）无动力游泳微生物的重量和速度。（c）具动力的游泳微生物。

A passive microswimmer does not have any self-propulsion. The center of mass G of the microswimmer is at a distance h and k from the midlines of the long and short arms respectively.
无动力的游泳微生物不具有任何自推进力。游泳微生物的质心 G 与长臂和短臂的中线的距离分别为 h 和 k。

B1	Write the expressions of h and k. 试写下 h 和 k 的表达式。	**2 points 2 分**

$h = \frac{b}{3}$, $k = \frac{4b}{3}$. [1,1]

The L-shaped microswimmer is tilted by an angle ϕ as shown in Fig. 2(b) and is sinking with velocity v in the direction inclined at an angle θ with the vertical in the presence of gravitational acceleration g. The microswimmer does not rotate. Assume that the upthrust of the fluid is negligible compared with the weight of the microswimmer.
L 形游泳微生物的倾斜角度为ϕ，如图 2（b）所示，并且在重力加速度 g 的影响下，以速度 v 下沉，速度相对垂直方向的倾斜角度为θ。游泳微生物不旋转。假设流体的浮力与游泳微生物的重量相比是可忽略的。

B2	Write the equation consisting of the components of all forces along the y axis (the direction of the long arm). 试写下沿 y 轴（长臂的方向）的所有分力组成的方程。	**2 points 2 分**

Friction on the long arm: $F_a = \mu 4bv\cos(\phi - \theta)/2$ [0.5]

Drag on the short arm: $D_b = \mu 2bv\cos(\phi - \theta)$ [0.5]

Hence $F_a + D_b = mg\cos\phi \Rightarrow 4\mu bv\cos(\phi - \theta) = mg\cos\phi$ [1]

B3	Write the equation consisting of the components of all forces along the x axis (direction of the short arm). 试写下沿 x 轴（短臂的方向）的所有分力组成的方程。	**2 points 2 分**

Drag on the long arm: $D_a = \mu 4bv\sin(\phi - \theta)$ [0.5]

Friction on the short arm: $F_b = \mu 2bv\sin(\phi - \theta)/2$ [0.5]

Hence $D_a + F_b = mg\sin\phi \Rightarrow 5\mu bv\sin(\phi - \theta) = mg\sin\phi$ [1]

B4	Write the equation consisting of the moments of all forces about the center of mass. 试写下所有围绕质心的力的力矩组成的方程。	**2 points 2 分**

Clockwise moments of the long arm in Fig. 2(b): [0.5]

Drag on the long arm: $D_a(2b - k) = 8\mu b^2 v\sin(\phi - \theta)/3$

Friction on the long arm: $F_a(a - h) = 2\mu b^2 v\cos(\phi - \theta)/3$

Anticlockwise moments of the short arm in Fig. 2(b): [0.5]

Drag on the short arm: $D_b(b - h) = 4\mu b^2 v\cos(\phi - \theta)/3$

Friction on the short arm: $F_b(2b - k) = 4\mu b^2 v\sin(\phi - \theta)/3$

Hence $D_a(2b - k) + F_a(a - h) = D_b(b - h) + F_b(2b - k)$

$\Rightarrow 8\mu b^2 v\sin(\phi - \theta)/3 + 2\mu b^2 v\cos(\phi - \theta)/3 = 4\mu bv\cos(\phi - \theta)/3 + 4\mu bv\sin(\phi - \theta)/3$ [1]

$$\tan(\phi - \theta) = \frac{1}{2}$$

B5	Calculate the tilt angle ϕ of the microswimmer at the steady state. Give your answer in degrees. 试计算游泳微生物在稳态下的倾斜角 ϕ。答案以度数表达。	**1 point 1 分**

From B2 and B3,

$mg\cos\phi = 4\mu bv\cos(\phi - \theta)$

$mg\sin\phi = 5\mu bv\cos(\phi - \theta)$

Dividing, $\tan\phi = \frac{5}{4}\tan(\phi - \theta) = \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) = \frac{5}{8} \Rightarrow \phi = \arctan\frac{5}{8} = 0.56\ \text{rad} = 32^0$ [1]

B6	Calculate the motion direction θ of the microswimmer at the steady state. Give your answer in degrees.	**1 point**

	试计算游泳微生物在稳态时的运动方向θ。答案以度数表达。	

$$\tan\theta = \frac{\tan\phi - \tan(\phi-\theta)}{1+\tan\phi\tan(\phi-\theta)} = \frac{5/8-1/2}{1+(5/8)(1/2)} = \frac{2}{21} \Rightarrow \theta = \arctan\frac{2}{21} = 0.095 \text{ rad} = 5.4^0$$ [1]

C. An Active Microswimmer with Rotation 具动力又会旋转的游泳微生物

To model an active microswimmer, physicists implemented a laser-induced chemical reaction at a point on the shorter arm of the object so that it provides a self-propulsion force F normal to the short arm. The dynamical properties of the microswimmer are rather sensitive to the point of application of F. For convenience we consider the case that this point is located at a distance $l_f = \frac{13}{24}b$ from the corner (see Fig. 2(c)). The force can be adjusted by tuning the laser intensity incident on the microswimmer. Note that it is possible that the microswimmer can rotate so that forces and torques due to rotation have to be included. The velocity v, direction θ and the tilt angle ϕ becomes time dependent, and you will need to include the angular velocity $\dot{\phi}$ as one of the variables.

为了模拟具动力的游泳微生物，物理学家在物体短臂上的一点加进可由激光诱导的化学反应，为它提供垂直于短臂的自推进力 F。游泳微生物的动力学性质对于 F 的作用点是相当敏感的。为了方便起见，我们考虑这一点位于距离角落$l_f = \frac{13}{24}b$的情况（参见图 2（c））。调节射在游泳微生物上的激光强度，可以调节推进力。注意，因为游泳微生物可以旋转，我们必须加入考虑由于旋转引起的力和力矩。速度 v、方向θ和倾斜角ϕ变得与时间相关，你需要将角速度$\dot{\phi}$包括为其中一个变量。

C1	Write the equation consisting of the components of all forces along the y axis (the direction of the long arm). 试写下沿 y 轴（长臂的方向）的所有分力组成的方程。	**2 points 2 分**

Friction on the long arm due to translation: $F_{ta} = 2\mu bv\cos(\phi-\theta)$

Drag on the short arm due to translation: $D_{tb} = 2\mu bv\cos(\phi-\theta)$

Friction on the long arm due to rotation: $F_{ra} = \mu 4bh\dot{\phi}/2 = 2\mu b^2\dot{\phi}/3$

Drag on the short arm due to rotation: $D_{rb} = -\mu 2b(b-h)\dot{\phi} = -4\mu b^2\dot{\phi}/3$

Hence $F_{ta} + F_{ra} + D_{tb} + D_{rb} + F = mg\cos\phi \Rightarrow F + 4\mu bv\cos(\phi-\theta) - \frac{2}{3}\mu b^2\dot{\phi} = mg\cos\phi$

[1 for terms dependent on $\dot{\phi}$, 1 for the equation]

C2	Write the equation consisting of the components of all forces along the x axis (the direction of the short arm). 试写下沿 x 轴（短臂的方向）的所有分力组成的方程。	**2 points 2 分**

Drag on the long arm due to translation: $D_{ta} = 4\mu bv\sin(\phi-\theta)$

Friction on the short arm due to translation: $F_{tb} = \mu bv\sin(\phi - \theta)$

Drag on the long arm due to rotation: $D_{ra} = \mu 4b(2b-k)\dot{\phi} = 8\mu b^2\dot{\phi}/3$

Friction on the short arm due to rotation: $F_{rb} = -\mu 2bk\dot{\phi}/2 = -4\mu b^2\dot{\phi}/3$

Hence $D_{ta} + D_{ra} + F_{tb} + F_{rb} = mg\sin\phi \Rightarrow 5\mu bv\sin(\phi-\theta) + \frac{4}{3}\mu b^2\dot{\phi} = mg\sin\phi$

[1 for terms dependent on $\dot{\phi}$, 1 for the equation]

C3	Write the equation consisting of the moments of all forces about the center of mass. 试写下所有围绕质心的力的力矩组成的方程。	**4 points** **4 分**

Clockwise moments of the long arm in Fig. 2(c):

Drag on the long arm due to translation: $D_{ta}(2b-k) = 8\mu b^2 v\sin(\phi-\theta)/3$

Friction on the long arm due to translation: $F_{ta}(a-h) = 2\mu b^2 v\cos(\phi-\theta)/3$

Drag on the long arm due to rotation: $64\mu b^3\dot{\phi}/12 + D_{ra}(2b-k) = 64\mu b^3\dot{\phi}/9$ [0.5]

Friction on the long arm due to rotation: $F_{ra}h = 2\mu b^3\dot{\phi}/9$ [0.5]

Anticlockwise moments of the short arm in Fig. 2(c):

Drag on the short arm due to translation: $D_{tb}(b-h) = 4\mu b^2 v\cos(\phi-\theta)/3$

Friction on the short arm due to translation: $F_{tb}k = 4\mu b^2 v\sin(\phi-\theta)/3$

Drag on the short arm due to rotation: $-8\mu b^3/12 + D_{rb}(b-h) = -14\mu b^3\dot{\phi}/9$ [0.5]

Friction on the short arm due to rotation: $-F_{rb}k = -16\mu b^3\dot{\phi}/9$ [0.5]

Self-propulsion force: $F(13b/24 - h) = 5Fb/24$ [1]

$$8\mu b^2 v\sin(\phi-\theta)/3 + 2\mu b^2 v\cos(\phi-\theta)/3 + 64\mu b^3\dot{\phi}/9 + 2\mu b^3\dot{\phi}/9$$
$$= 4\mu b^2 v\cos(\phi-\theta)/3 + 4\mu b^2 v\sin(\phi-\theta)/3 - 14\mu b^3\dot{\phi}/9 - 16\mu b^3\dot{\phi}/9 + 5Fb/24$$

$$\frac{32}{3}\mu b^2\dot{\phi} = \frac{5}{24}F + \frac{2}{3}\mu bv\left[\cos(\phi-\theta) - 2\sin(\phi-\theta)\right]$$ [1]

The above three equations can be solved for the three variables $v\cos(\phi-\theta)$, $v\sin(\phi-\theta)$ and $\dot{\phi}$.

上述三个方程可以对 $v\cos(\phi-\theta)$、 $v\sin(\phi-\theta)$ 和 $\dot{\phi}$ 三个变量求解。

C4	Eliminate $v\cos(\phi-\theta)$ and $v\sin(\phi-\theta)$ from the above equations to obtain an equation involving ϕ and $\dot{\phi}$ only. 从上述方程中消去 $v\cos(\phi-\theta)$ 和 $v\sin(\phi-\theta)$ ，以获得一个单涉及 ϕ 和 $\dot{\phi}$ 的方程。	**2 points** **2 分**

From C1 to C3,

$$5\mu bv\sin(\phi-\theta) = mg\sin\phi - \frac{4}{3}\mu b^2\dot{\phi} \quad (1)$$

$$4\mu bv\cos(\phi-\theta) = mg\cos\phi + \frac{2}{3}\mu b^2\dot{\phi} - F \qquad (2)$$

$$\frac{32}{3}\mu b^2\dot{\phi} = \frac{5}{24}F + \frac{2}{3}\mu bv\left[\cos(\phi-\theta) - 2\sin(\phi-\theta)\right] \qquad (3)$$

Substituting (1) and (2) into (3),

$$\frac{153}{15}\mu b^2\dot{\phi} = \frac{1}{24}F - \frac{mg}{30}(8\sin\phi - 5\cos\phi)$$

$$\dot{\phi} = \frac{1}{306\mu b^2}\left[\frac{5F}{4} - mg(8\sin\phi - 5\cos\phi)\right] = \frac{1}{306\mu b^2}\left[\frac{5F}{4} - \sqrt{89}\,mg\sin(\phi-\phi_0)\right] \text{ where } \sin\phi_0 = \frac{5}{\sqrt{89}}$$

[1 point for steps, 1 point for the result]

C5	Derive the tilt angle ϕ when the microswimmer reaches the steady state of constant tilt. 试推导游泳微生物在固定倾斜稳态下的倾斜角ϕ。	**2 points** **2 分**

$$\frac{1}{306\mu b^2}\left[\frac{5F}{4} - \sqrt{89}\,mg\sin(\phi-\phi_0)\right] = 0 \Rightarrow \phi = \arcsin\frac{5}{\sqrt{89}} + \arcsin\frac{5F}{4\sqrt{89}\,mg}$$

C6	Consider a microswimmer initially at the steady state with $F = 0$. At $t = 0$ the laser is switched on so that F becomes nonzero. Calculate $\phi(t)$ for $F \ll mg$. 考虑游泳微生物的初始状态处於 $F = 0$ 的稳态。在 $t = 0$ 时，激光亮了，使得 F 变为非零。在 $F \ll mg$ 的情况下，试计算$\phi(t)$。	**2 points** **2 分**

$$\frac{d\phi}{dt} = \frac{1}{306\mu b^2}\left[\frac{5F}{4} - \sqrt{89}\,mg\sin(\phi-\phi_0)\right] \approx -\frac{\sqrt{89}\,mg}{306\mu b^2}\left(\phi - \phi_0 - \frac{5F}{4\sqrt{89}\,mg}\right) \qquad [1]$$

Solution: $$\phi = \phi_0 + \frac{5F}{4\sqrt{89}\,mg}\left[1 - \exp\left(-\frac{\sqrt{89}\,mg}{306\mu b^2}t\right)\right] \qquad [1]$$

When F gradually increases from 0, the direction of linear motion gradually changes. When F exceeds a critical value, the tilt angle is no longer constant and the microswimmer takes a wheel-like trajectory.

当 F 从 0 渐渐增加时，游泳微生物线性运动的方向渐渐改变。当 F 超过临界值时，倾斜角不再恒定，游泳微生物的轨迹变成轮状。

C7	Write the maximum value of F for linear motion. 写下线性运动的最大 F 值。	**1 point** **1 分**

The equation $\frac{1}{306\mu b^2}\left[\frac{5F}{4}-\sqrt{89}\,mg\,\sin(\phi-\phi_0)\right]=0$ has no solution when $F>\frac{4\sqrt{89}}{5}mg$. Hence the answer is $F=\frac{4\sqrt{89}}{5}mg$. [1]

C8	To verify that the microswimmer can move in a wide range of directions, calculate the force(s) required for linear motion in the horizontal direction $\theta = \pi/2$. Give your answer in multiples of mg to 3 significant figures. 为了验证游泳微生物能够在广阔范围内的方向移动，试计算游泳微生物在水平方向$\theta = \pi/2$ 作线性运动时，所需的(诸)力是多少。答案以 mg 的倍数表达，至 3 位有效数字。	**3 points** **3 分**

From C4, $\dot{\phi}=0 \Rightarrow F=\frac{4}{5}mg(8\sin\phi-5\cos\phi)$

From C1 and C2,

$$5\mu bv\sin(\phi-\theta)=mg\sin\phi$$
$$4\mu bv\cos(\phi-\theta)=mg\cos\phi-F$$ [1]

Dividing, $\tan(\phi-\theta)=\frac{4mg\sin\phi}{5(mg\cos\phi-F)}=\frac{4\sin\phi}{25\cos\phi-32\sin\phi}$

When $\theta=\frac{\pi}{2}$, $\tan(\phi-\theta)=-\cot\phi=-\frac{\cos\phi}{\sin\phi} \Rightarrow -\frac{\cos\phi}{\sin\phi}=\frac{4\sin\phi}{25\cos\phi-32\sin\phi}$

$$25\cos^2\phi-32\sin\phi\cos\phi+4\sin^2\phi=0$$ [1]

Since $\cos^2\phi=\frac{1}{2}+\frac{1}{2}\cos 2\phi$, $\sin^2\phi=\frac{1}{2}-\frac{1}{2}\cos 2\phi$, $\sin\phi\cos\phi=\frac{1}{2}\sin 2\phi$,

$$32\sin 2\phi-21\cos 2\phi=29$$

$\sqrt{1465}\sin(2\phi-2\phi_1)=29$ where $\sin 2\phi_1=\frac{21}{\sqrt{1465}}$

$$\phi=\frac{1}{2}\left(\arcsin\frac{29}{\sqrt{1465}}+\arcsin\frac{21}{\sqrt{1465}}\right)=0.7202 \Rightarrow F=\frac{4}{5}mg(8\sin\phi-5\cos\phi)=1.21\,mg$$

$$\phi=\frac{1}{2}\left(\pi-\arcsin\frac{29}{\sqrt{1465}}+\arcsin\frac{21}{\sqrt{1465}}\right)=1.4313 \Rightarrow F=\frac{4}{5}mg(8\sin\phi-5\cos\phi)=5.78\,mg$$

[1 for either result]

The background material of this problem can be found in the article: B. ten Hagen, F. Kümmel, R. Wittkowski, D. Takagi, H. Löwen, and C. Bechinger, "Gravitaxis of asymmetric self-propelled colloidal particles", Nature Communications **5**, 4829 (2014).