# Problem 1: Spatiotemporal varying electric permittivity (30 points)

# 问题 1: 时变介电常数 (30 分)

In electromagnetism, dielectric media, e.g. a block of glass, is represented by a permittivity different from the one of vacuum. The reflection and refraction for electromagnetic waves on a slab of dielectric material can be derived from considering dispersion relationship and matching boundary condition.

在电磁学中，不同于真空介质，例如一块玻璃，其介电材料可以用特定的介电常数表示。在给定色散关系和匹配边界条件下，可以导出电磁波在介电材料板上的反射和折射。

In the question, we would like to investigate the Fabry-Pérot resonance from a slab of dielectric medium in the first step and the analog concept when the permittivity of the material becomes inhomogeneous in the time domain instead of the spatial domain.

在这个问题中，我们第一步研究电介质板的法布里-珀罗共振现象，以及当材料的介电常数在时域而不是空间域中变得不均匀时类似的概念。

Figure 1 shows the schematic diagrams for light entering a block of dielectric medium at normal incidence. The left diagram shows the case for a block of infinite thickness, in which the light only undergoes one instance of reflection and refraction at the interface. The right diagram shows the case for a finite thickness, in which the light ray undergoes multiple reflections within the slab. The spatial and temporal axes are the horizontal and vertical ones respectively.

图 1 显示了光线以垂直入射方式进入电介质板的示意图。左图显示了无限厚度板的情况，其中光在界面处仅经历一次反射和折射。右图显示了有限厚度板的情况，其中光线在板内经历了多次反射。空间轴和时间轴分别是图中水平轴和垂直轴。

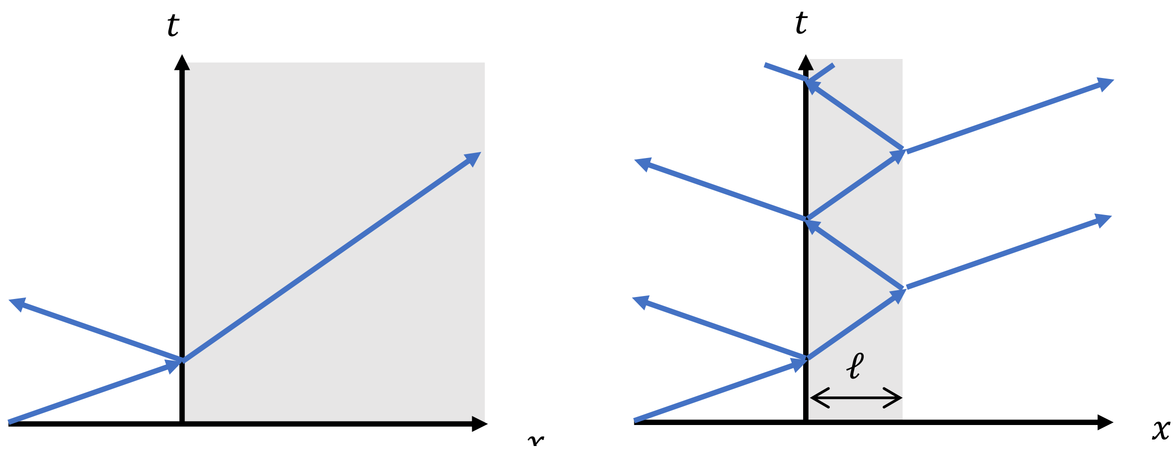


Fig. 1: (Left) Schematic diagram for a light ray entering an interface between vacuum and dielectric medium (Right) Schematic diagram for a light ray entering a slab of dielectric medium undergoing multiple reflections.

图 1：（左）光线进入真空和电介质的示意图，（右）光线进入电介质经过多次反射的示意图。

**A. Fabry-Pérot resonance (13 points)**

To obtain the amount of light being reflected and refracted from a block of dielectric medium, we start from the Maxwell's equations:

为了获得光经过一块电介质板后的反射和折射强度，我们从麦克斯韦方程开始：

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t},$$

where the curl and div operators are defined by

其中 curl 运算符定义为

$$\nabla \times \boldsymbol{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y}\right)$$

The constitutive relationship for the dielectric material or vacuum in relating the different field variables are expressed as

介电材料或真空与不同场变量相关的本构关系为

$$\boldsymbol{D} = \epsilon(x)\boldsymbol{E}$$
$$\boldsymbol{B} = \mu_0\boldsymbol{H}$$

In this part, the electric permittivity profile $\epsilon(x)$ only varies in $x$ but not in $y, z$ and $t$. All the materials are non-magnetic, having the same value of magnetic permeability $\mu_0$ of vacuum. We only consider electromagnetic waves which propagate in the $x$-direction.

在本部分中，介电常数曲线 $\epsilon(x)$ 仅随着 $x$ 变化，而与 $y$、$z$ 和 $t$ 无关。 所有材料都是非磁性的，具有相同的真空磁导率$\mu_0$值。 我们只考虑在 $x$ 方向传播的电磁波。

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| A1 | We consider an electromagnetic wave propagating only in the x-direction specified by $\boldsymbol{E} = \hat{y}E_y(x,t)$, $\boldsymbol{H} = \hat{z}H_z(x,t)$. Reduce the Maxwell's equations to two differential equations only on $E_y$ and $H_z$. Each differential equation is first order in both temporal dimension $t$ and spatial dimension $x$.<br><br>我们仅考虑由$\boldsymbol{E} = \hat{y}E_y(x,t)$，$\boldsymbol{H} = \hat{z}H_z(x,t)$指定的在$x$方向传播的电磁波。请将麦克斯韦方程组简化为$E_y$和$H_z$上的两个微分方程式。每个分方程式在时间维度 $t$ 和空间维度 $x$ 上都具有一阶。 | 2 Points<br>2 分 |

Answer:

$$\partial_x E_y = -\partial_t B_z = -\mu_0 \partial_t H_z$$
$$\partial_x H_z = -\partial_t D_y = -\epsilon(x)\partial_t E_y$$

| A2 | For a sinusoidal wave propagating in the positive *x*-direction in a medium of constant permittivity $\epsilon_1$ (e.g. in a glass), it has a form $E_y = E_{in}\cos(k_1 x - \omega t)$. Find the dispersion relationship between $k_1$ and $\omega$<br><br>对于在介电常数为 $\epsilon_1$ 的介质（例如在玻璃中）中沿 $x$ 正方向传播的正弦波，它的形式为 $E_y = E_{in}\cos(k_1 x - \omega t)$。 求$k_1$与 ω 的色散关系。 | 2 Points<br>2 分 |
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Answer:

Let also $H_z = H_{in}\cos(k_1 x - \omega t)$, we have

$$-k_1 E_y = -\omega\mu_0 H_z$$
$$-k_1 H_z = -\omega\epsilon_1 E_y$$
$$\Rightarrow k_1 = \omega\sqrt{\mu_0\epsilon_1}, \qquad H_z = \frac{k_1}{\omega\mu_0}E_y = \frac{\sqrt{\epsilon_1}}{\sqrt{\mu_0}}E_y \triangleq \frac{E_y}{\eta_1}$$

Now, we consider an interface between vacuum and a dielectric medium (see left panel of Fig. 1). A fixed incident electromagnetic wave $E_y = E_{in}\cos(k_0 x - \omega t)$ propagates in vacuum to the right hand side before entering the dielectric medium. The permittivity $\epsilon(x)$ in this case is a step function, being $\epsilon_0$ for $x < 0$ and $\epsilon_1$ for $x \geq 0$.

现在，我们考虑真空和电介质之间的界面(见图 1 的左图)。一个固定的入射电磁波 $E_y = E_{in}\cos(k_0 x - \omega t)$ 在进入玻璃之前在真空中传播到右手边。在这种情况下，介电常数 $\epsilon(x)$ 是一个阶跃函数，当$x < 0$ 为 $\epsilon_0$，当 $x \geq 0$ 为 $\epsilon_1$。

| A3 | Suppose the transmitted waves is represented by $E_y = E_t\cos(k_1 x - \omega t + \phi_t)$. Find $E_t$ and $\phi_t$ in terms of $E_{in}$, $\omega$ and the permittivities $\epsilon_0$, $\epsilon_1$<br><br>假设传输的波由 $E_y = E_t\cos(k_1 x - \omega t + \phi_t)$ 表示。 根据 $E_{in}$、ω 和其他介电常数 $\epsilon_0, \epsilon_1$，求 $E_t$ 和 $\phi_t$ | 5 Points<br>5 分 |
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Answer:

Let also the reflected wave as $E_y = E_r\cos(k_0 x - \omega t + \phi_r)$, $k_i = \omega\sqrt{\mu_0\epsilon_i}$ , $\eta_i = \sqrt{\mu_0}/\sqrt{\epsilon_i}$.
For $x \le 0$,

$$E_y = E_{in}\cos(k_0 x - \omega t) + E_r\cos(-k_0 x - \omega t + \phi_r)$$
$$H_z = \frac{E_{in}}{\eta_0}\cos(k_0 x - \omega t) - \frac{E_r}{\eta_0}\cos(-k_0 x - \omega t + \phi_r)$$

For $x \ge 0$,

$$E_y = E_t\cos(k_1 x - \omega t + \phi_t)$$
$$H_z = \frac{E_t}{\eta_1}\cos(-k_1 x - \omega t + \phi_t)$$

On the boundary, the continuous variables are $E_y$ and $H_z$

$$E_{in}\cos(\omega t) + E_r\cos(\omega t - \phi_r) = E_t\cos(\omega t - \phi_t)$$
$$\frac{E_{in}}{\eta_0}\cos(\omega t) - \frac{E_r}{\eta_0}\cos(\omega t - \phi_r) = \frac{E_t}{\eta_1}\cos(\omega t - \phi_t)$$

Expand into $\cos(\omega t)$ and $\sin(\omega t)$, we obtain the following equations:

$$E_{in} + E_r\cos\phi_r = E_t\cos\phi_t$$
$$(E_{in} - E_r\cos\phi_r)/\eta_0 = E_t\cos\phi_t/\eta_1$$
$$E_r\sin\phi_r = E_t\sin\phi_t$$
$$-E_r\sin\phi_r/\eta_0 = E_t\sin\phi_t/\eta_1$$

Therefore,

$$E_t = \frac{2\eta_1}{\eta_0 + \eta_1}E_{in}$$
$$E_r = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0}E_{in}$$
$$\phi_r = \phi_t = 0$$

We can also define transmission and reflection coefficients $t$ and $r$ by

$$t = \frac{2\eta_1}{\eta_0 + \eta_1} = \frac{2\sqrt{\epsilon_0}}{\sqrt{\epsilon_0} + \sqrt{\epsilon_1}}$$
$$r = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} = \frac{\sqrt{\epsilon_0} - \sqrt{\epsilon_1}}{\sqrt{\epsilon_0} + \sqrt{\epsilon_1}}$$

Now, we consider a plate of dielectric medium with a finite thickness $\ell$, with a fixed electromagnetic wave shining normally to the plate. There will be multiple reflections and transmissions between the two interfaces of the dielectric medium (see right panel of Fig. 1). The incident wave is still fixed as $E_y = E_{in}\cos(k_0 x - \omega t)$ while the total transmitted waves after the whole slab of dielectric medium is now specified by $E_y = E_t^{(\text{tot})}\cos\left(k_0 x - \omega t + \phi_t^{(\text{tot})}\right)$, by summing all the multiple reflections and transmissions.

现在，我们考虑具有有限厚度 $l$ 的电介质板，即一个固定的电磁波正入射到电介质板。在电介质的两个界面之间会有多次反射和透射(如图 1 的右图)。入射波仍然固定为

$E_y = E_{in}\cos(k_0 x - \omega t)$，而整个电介质板之后的透射波可以写成为 $E_y = E_t^{(\text{tot})}\cos\left(k_0 x - \omega t + \phi_t^{(\text{tot})}\right)$，通过对所有多次反射和透射求和。

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| A4 | Derive the maximum value for $\left\|E_t^{(\text{tot})}\right\|$ when we can choose an optimal value of $\ell$ . Derive the condition for such $\ell$.<br><br>当我们可以选择 $\ell$ 的最佳值时，导出 $\left\|E_t^{(\text{tot})}\right\|$ 的最大值以及$\ell$满足的条件。 | 4 Points<br>4 分 |

Answer:

From last part, it is actually more convenient to use complex number notation, e.g.

$$E_r\cos\phi_r + iE_r\sin\phi_r \to e_r$$

Then

$$e_t = \frac{2\eta_1}{\eta_0+\eta_1}e_{in}$$
$$e_r = \frac{\eta_1-\eta_0}{\eta_1+\eta_0}e_{in}$$

From vacuum to dielectric medium:

$$t = \frac{2\eta_1}{\eta_0+\eta_1}, \qquad r = \frac{\eta_1-\eta_0}{\eta_1+\eta_0}$$

From dielectric medium to vacuum:

$$t' = \frac{2\eta_0}{\eta_0+\eta_1}, \qquad r' = \frac{\eta_0-\eta_1}{\eta_1+\eta_0} = -r$$

Total transmission (summing the multiple scattering processes in right panel of Fig. 1) is

$$t_{tot} = te^{ik_1\ell}t' + te^{ik_1\ell}r'\,e^{ik_1\ell}r'\,e^{ik_1\ell}t' + \cdots$$
$$= \frac{tt'e^{ik_1\ell}}{1-r'^2e^{2ik_1\ell}}$$

Therefore, the maximum value of $\left|E_t^{(\text{tot})}\right|$ is

$$\frac{4\eta_0\eta_1}{(\eta_0+\eta_1)^2}\frac{1}{1-\left(\frac{\eta_0-\eta_1}{\eta_1+\eta_0}\right)^2} = \frac{4\eta_0\eta_1}{4\eta_0\eta_1} = 1$$

The condition is

$$k_1\ell = m\pi$$

for an arbitrary integer $m$.

**B. Time-varying permittivity (14 points)**

We have considered a boundary of the permittivity profile in the spatial domain $x$. Now we go to the complementary problem. Suppose the electric permittivity is constant in all spatial dimensions but varies in temporal dimension $t$. For example, materials like $\mathrm{LiNbO_3}$ will have its permittivity changes when a DC voltage is applied across the material through the Kerr electro-optic effect.

我们已经考虑了空间域 x 中介电常数分布的边界。 现在我们转到其互补问题。 假设介电常数在所有空间维度上都是恒定的，但在时间维度 t 上变化。例如，当通过克尔电光效应在材料上施加直流电压时，LiNb03 等材料的介电常数会发生变化。

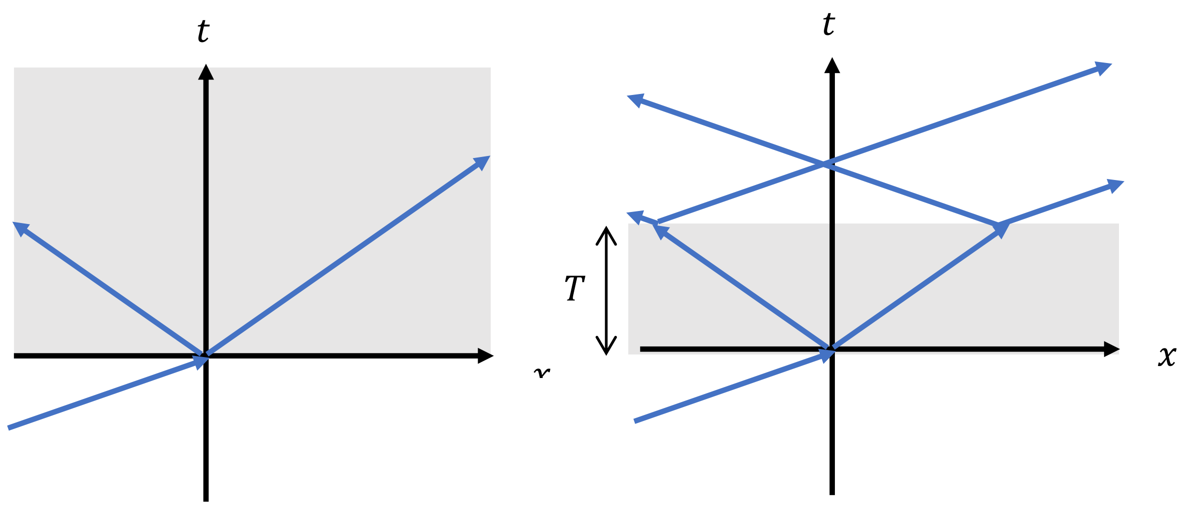


Fig. 2: (Left) Schematic diagram for a light ray entering a time-boundary in which the permittivity changes from $\epsilon_1$ to $\epsilon_2$ (Right) Schematic diagram for a light ray going through two time-boundaries. The permittivity changes from $\epsilon_1$ to $\epsilon_2$ at $t = 0$ and changes back to $\epsilon_1$ at $t = T$.

图 2：（左）光线进入时间边界的示意图，其中介电常数从 $\epsilon_1$ 变为 $\epsilon_2$（右）光线穿过两个时间边界的示意图。 介电常数在 t=0 时从 $\epsilon_1$ 变为 $\epsilon_2$，并在 t=T 时变回 $\epsilon_1$。

In a similar fashion, left panel of Fig. 2 shows the trajectory for a light ray originally propagating in a homogeneous medium in the positive direction. Then, the permittivity suddenly changes from $\epsilon_1$ to another value $\epsilon_2$ at $t = 0$. In the right panel, we now have a “slab” of such time-varying material in the time domain. The permittivity changes back to $\epsilon_1$ at $t = T$. The blue arrows show the light rays. At every boundary, the ray splits in a forward and backward propagating ray. As the initial ray is propagating to the right, we call the left propagating wave as “reflection/backward propagating” and the right propagating wave as “transmission/forward propagating” after a time boundary. Note that the rays can only to in the positive time direction.

以类似的方式，图 2 的左图显示了最初在一个均质介质中沿正方向传播的光线的轨迹。然后，介电常数在 t=0 时突然从 $\epsilon_1$ 变为另一个值 $\epsilon_2$。 在右侧面板中，我们在时域中有一个这种随时间变化的材料的“板”。 介电常数在 t=T 时变回 $\epsilon_1$。 蓝色箭头表示光线。 在每个边界处，光线分裂为向前和向后传播的光线。 由于初始光线向右传播，我们称左传播波为“反射/后向传播”，右传播波在时间边界后称为“透射/前向传播”。请注意，光线只能在正时间方向上前进。

To be specific, consider an electromagnetic waves propagates in an infinite block of dielectric medium originally. Suddenly, at time equal to zero, the dielectric constant in the whole space changes from $\epsilon_1$ to $\epsilon_2$. We call $t = 0$ as a temporal boundary. See left panel of Fig. 2.

具体来说，考虑电磁波最初在无限大的电介质中传播。 突然，在时间等于 0 时，在整个空间介电常数从 $\epsilon_1$ 变为 $\epsilon_2$。 我们称 t=0 为时间边界。参见图 2 的左侧。

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| B1 | What are the continuity conditions in this case across the time boundary?<br><br>在这种情况下，跨越时间边界的连续性条件是什么？ | 3 Points<br>3 分 |

Answer:

From the Maxwell’s equations,

$$\partial_x E_y = -\partial_t B_z$$
$$\partial_x H_z = -\partial_t D_y$$

under the similar principle to consider a spatial boundary, a temporal boundary requires continuity of $D_y$ and $B_z$.

Initially, the electromagnetic waves is specified by $\boldsymbol{E} = \hat{y}E_1\cos(k_1x - \omega t)$, a forward propagating waves in the positive *x* direction. After the permittivity has changed, there is a backward propagating waves (along the negative x direction) and forward propagating waves (along the positive *x* direction). They are the defined as the temporal reflection and temporal transmission waves.

最初，电磁波由 $\boldsymbol{E} = \hat{y}E_1\cos(k_1x - \omega t)$ 指定，即在正 *x* 方向上的前向传播波。介电常数改变后，有向后传播的波（沿负 x 方向）和向前传播的波（沿正 x 方向）。 它们被定义为时间反射波和时间传输波。

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| B2 | The radial frequency has changed from $\omega$ to another value $\omega'$ across the time-boundary. Express $\omega'$ in terms of $\omega$ and the permittivity values.<br><br>径向频率在时间边界上已从 ω 变为另一个值 ω'。 用 ω 和介电常数值表示 ω'。 | 3 Points<br>3 分 |

Answer:

For $t \le 0$,

$$E_y = E_1\cos(k_1x - \omega t)$$
$$H_z = \frac{E_1}{\eta_1}\cos(k_1x - \omega t)$$

For $t \ge 0$,

$$E_y = E_r\cos(k_2x - \omega' t - \phi_r) + E_t\cos(k_2x - \omega' t + \phi_t)$$
$$H_z = -\frac{E_r}{\eta_2}\cos(k_2x - \omega' t - \phi_r) + \frac{E_t}{\eta_2}\cos(k_2x - \omega' t + \phi_t)$$

where $k_2 = \omega'\sqrt{\mu_0\epsilon_2}$.

Continuity gives

$$\epsilon_1E_1\cos(k_1x) = \epsilon_2E_r\cos(k_2x - \phi_r) + \epsilon_2E_t\cos(k_2x + \phi_t)$$
$$\frac{E_1}{\eta_1}\cos(k_1x) = -\frac{E_r}{\eta_2}\cos(k_2x - \phi_r) + \frac{E_t}{\eta_2}\cos(k_2x + \phi_t)$$

The condition can only be satisfied by setting $k_2 = k_1$. This is in analogy to the spatial case that $\omega$ is conserved across a spatial boundary. For a time-boundary. $k$ is conserved.

Therefore

$$\omega' = \sqrt{\epsilon_1}\omega/\sqrt{\epsilon_2}$$

From now on, I just call $k = k_1 = k_2 = \omega\sqrt{\mu_0\epsilon_1}$

| B3 | Express the amplitude of the backward propagating wave and the forward propagating wave.<br><br>求后向传播波和前向传播波的幅度。 | 3 Points<br>3 分 |
|---|---|---|

Answer:

From last part, the continuity is derived as

Continuity gives

$$\epsilon_1 E_1 \cos(kx) = \epsilon_2 E_r \cos(kx - \phi_r) + \epsilon_2 E_t \cos(kx + \phi_t)$$
$$\frac{E_1}{\eta_1}\cos(kx) = -\frac{E_r}{\eta_2}\cos(kx - \phi_r) + \frac{E_t}{\eta_2}\cos(kx + \phi_t)$$

We therefore have $\phi_t = \phi_r = 0$ and

$$\epsilon_1 E_1 = \epsilon_2 E_r + \epsilon_2 E_t$$
$$\frac{E_1}{\eta_1} = \frac{E_r}{\eta_2} + \frac{E_t}{\eta_2}$$

So, we have

$$E_r = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right)E_1 = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\right)E_1$$
$$E_t = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right)E_1 = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\right)E_1$$

Suppose the whole medium goes back to a dielectric medium of permittivity of $\epsilon_1$ after time $t = T$ (see right hand side of Fig. 2).

假设在时间 $t = T$ 之后，整个介质回到介电常数为 $\epsilon_1$ 的电介质 （参见图 2 的右侧）。

| B4 | Find out the amplitude of the backward and forward propagating waves finally. What is the condition to get minimal reflection?<br><br>最后求出后向传播波和前向传播波的幅度。 获得最小反射的条件是什么？ | 5 Points<br>5 分 |
|---|---|---|

Answer:

We use a tuple to represent the backward and forward propagating waves and use complex quantities. The different processes (see right hand side of Fig. 2) can be written as the following step by step.

At $t = 0$

$$(0,1) \rightarrow \left(\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right), \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right)\right)$$

At $t = T^-$, the two waves propagate and become

$$\left(\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right) e^{-i\omega' T}, \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right) e^{i\omega' T}\right)$$

At $t = T$, we have two processes:

$$(0,1) \rightarrow \left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right)\right)$$

$$(1,0) \rightarrow \left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right)\right)$$

As the system is linear, the final waves become

$$\left(\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right) e^{-i\omega' T}, \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right) e^{i\omega' T}\right)$$

$$\rightarrow \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \frac{\eta_2}{\eta_1}\right) e^{-i\omega' T}\left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right)\right)$$

$$+ \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\eta_2}{\eta_1}\right) e^{i\omega' T}\left(\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} + \frac{\eta_1}{\eta_2}\right), \frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1} - \frac{\eta_1}{\eta_2}\right)\right)$$

$$= \left(\frac{i}{2}\sin(\omega' T)\left(-\frac{\epsilon_1\eta_1}{\epsilon_2\eta_2} + \frac{\epsilon_2\eta_2}{\epsilon_1\eta_1}\right), \cos(\omega' T) + \frac{i}{2}\sin(\omega' T)\left(\frac{\epsilon_1\eta_1}{\epsilon_2\eta_2} + \frac{\epsilon_2\eta_2}{\epsilon_1\eta_1}\right)\right)$$

$$= \left(\frac{i}{2}\sin(\omega' T)\left(-\frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} + \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\right), \cos(\omega' T) + \frac{i}{2}\sin(\omega' T)\left(\frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} + \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\right)\right)$$

The condition to get minimal reflection is $\omega' T = m\pi$.

**C. Spatiotemporal permittivity (5 points)**

In the final part, we consider the case that the permittivity has to be written as a function in both $x$ and $t$, i.e. $\epsilon(x,t)$. For a simple case, the permittivity function consists of two constants: $\epsilon_1$ in the white region and $\epsilon_2$ in the gray region, a so-called spatiotemporal boundary depicted in Fig. 3. The spatiotemporal boundary is defined by a straight line $x = -ut$

在最后一部分，我们考虑介电常数必须写为 x 和 t 的函数的情况，即 ϵ(x,t)。 对于一个简单的情况，介电常数函数由两个常数组成：白色区域中的 $\epsilon_1$ 和灰色区域中的 $\epsilon 2$，即所谓的时空边界，如图 3 所示。时空边界由直线 $x = -ut$ 定义。

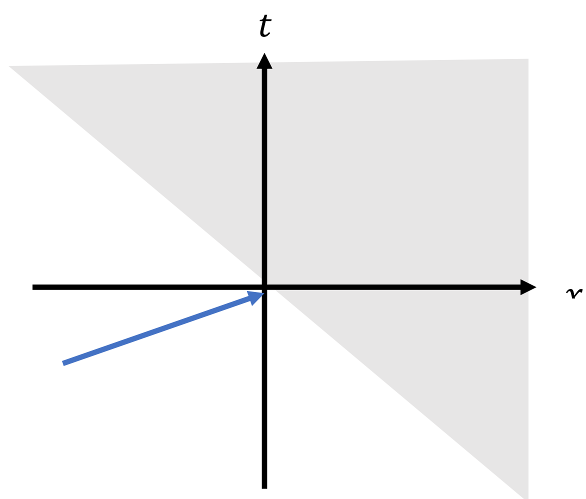



Fig. 3: A spatiotemporal boundary. In the gray (white) region, the permittivity is $\epsilon_1$ ($\epsilon_2$)

图 3：时空边界。 在灰色（白色）区域，介电常数为 $\epsilon_1$ ($\epsilon_2$)

| | | |
|---|---|---|
| C1 | What are the continuity conditions in this case across the spatiotemporal boundary? Hint: You may need to consider a coordinate transformation.<br><br>在这种情况下，跨越时空边界的连续性条件是什么？提示：您可能需要考虑坐标变换。 | 3 Points<br>3 分 |

Answer:

We change the coordinate from $(x, t)$ to $(q, \tau)$

$$q = q(x, t)$$
$$\tau = x + ut$$

As $\epsilon(x, t) = \epsilon(\tau)$, we would like to observe how the Maxwell's equations connect field quantities at different $\tau$.

$$\partial_x = \frac{\partial q}{\partial x}\partial_q + \partial_\tau$$
$$\partial_t = \frac{\partial q}{\partial x}\partial_q + u\partial_\tau$$

From the Maxwell's equations,

$$\partial_x E_y + \partial_t B_z = 0$$
$$\partial_x H_z + \partial_t D_y = 0$$

we have

$$\frac{\partial q}{\partial x}\left(\partial_q E_y + \partial_q B_z\right) + \partial_\tau\left(E_y + uB_z\right) = 0$$
$$\frac{\partial q}{\partial x}\left(\partial_q H_z + \partial_q D_y\right) + \partial_\tau\left(H_z + uD_y\right) = 0$$

The continuity variables are

$$E_y + uB_z$$
$$H_z + uD_y$$

being continuous across the boundary

A more specific way:

Define $u = c_1 \cot\theta$ so that the boundary is described by

$$c_1 \cos\theta \ t + \sin\theta \ x = 0$$

We change coordinate from $(x, t)$ to fictitious $(x', t')$ by

$$\begin{pmatrix} x' \\ c_1 t' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ c_1 t \end{pmatrix}$$

$$\begin{pmatrix} \partial_x \\ \frac{1}{c_1}\partial_t \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \partial_{x'} \\ \frac{1}{c_1}\partial_{t'} \end{pmatrix}$$

From the Maxwell's equations,

$$\partial_x E_y + \partial_t B_z = 0$$
$$\partial_x H_z + \partial_t D_y = 0$$

$$\cos\theta\,\partial_{x'} E_y + \sin\theta \frac{1}{c_1}\partial_{t'} E_y + c_1\left(-\sin\theta\,\partial_{x'} B_z + \cos\theta \frac{1}{c_1}\partial_{t'} B_z\right) = 0$$
$$\cos\theta\,\partial_{x'} H_z + \sin\theta \frac{1}{c_1}\partial_{t'} H_z + c_1\left(-\sin\theta\,\partial_{x'} D_y + \cos\theta \frac{1}{c_1}\partial_{t'} D_y\right) = 0$$

$$\partial_{x'}(\cos\theta\, E_y - \sin\theta\, c_1 B_z) + \partial_{t'}\left(\sin\theta \frac{1}{c_1} E_y + \cos\theta\, B_z\right) = 0$$
$$\partial_{x'}(\cos\theta\, H_z - \sin\theta\, c_1 D_y) + \partial_{t'}\left(\sin\theta \frac{1}{c_1} H_z + \cos\theta\, D_y\right) = 0$$

Since $\epsilon$ is a function of $t'$ and independent of $x'$,

The continuity variables are

$$\sin\theta \frac{1}{c_1} E_y + \cos\theta\, B_z$$
$$\sin\theta \frac{1}{c_1} H_z + \cos\theta\, D_y$$

or equivalently

$$E_y + uB_z$$
$$H_z + uD_y$$

being continuous across the boundary

# Problem 2: Fractal Dimensions of Networks (30 points)
# 问题 2: 网络的分数维数 (30 分)

A line is one-dimensional, a plane is two-dimensional, and the volume of a ball is three-dimensional. Is the dimension of an object always an integer? What's the dimension of a coastline? What's the dimension of the Internet or social network? The dimension of complicated objects is an interdisciplinary study between physics and many other sciences. Here, we will use "box counting dimension" to discuss the dimension of fractal and complex network objects.

直线是一维物体，平面是二维的，而球体是三维的。物体的维度总是整数吗？海岸线是多少维的，互联网、朋友圈又是多少维的？复杂物体的维度，是物理学和多个学科的交叉学科。这里，我们将使用“计盒维数”讨论分形几何和复杂网络的维度问题。

Based on this, we discuss the "renormalization group" of a complex network. Renormalization group describes how physics theories vary as a function of scales, and thus is "theory of theory". Renormalization group is first discovered in the quantum field theory of high energy particle physics and condensed matter physics. Here, complex networks made by vertices and edges, is probably the simplest example to introduce renormalization group.

以此为基础，我们讨论复杂网络的“重整化群”问题。重整化群是物理理论随着尺度变化而变化的现象，是“理论中的理论”。重整化群在高能粒子物理和凝聚态物理的量子场论描述中最先被发现。而只由点和线组成的复杂网络，或许是介绍重整化群的最简单的例子。

## PART A. FRACTALS AND BOX COUNTING DIMENSIONS 分形和计盒维数

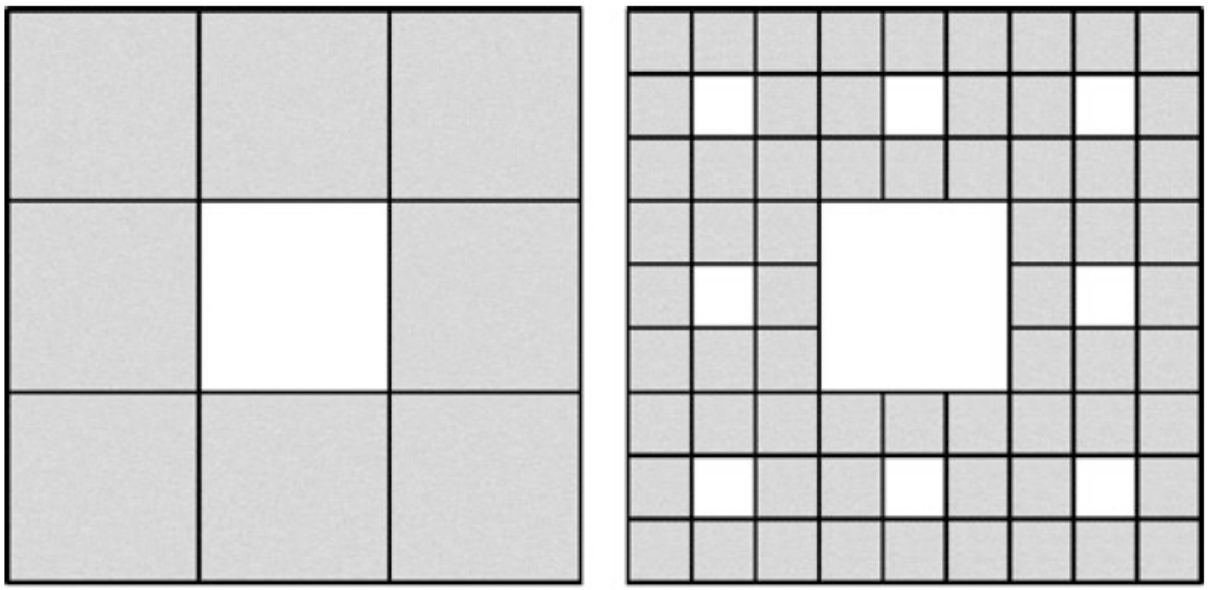
We cover an image using squares with side length no more than $s$. Let $B(s)$ be the least number of squares to cover the whole image. Then the box counting dimension $d_B$ is

我们用边长不超过 $s$ 的正方形来覆盖整个图形。设 $B(s)$ 是可以覆盖整个图形所用正方形的最少数目。则计盒维数 $d_B$ 为

$$d_B \equiv \lim_{s\to 0}\frac{\log B(s)}{\log(1/s)} .$$

In this problem we assume the following limit exists. For example, the Sierpinski carpet is the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations). Through infinite iteration, self-similar patterns emerge in the following figure. Such complex objects is known as fractals. In each iteration, in the nine squares, the center one is removed. What's the box counting dimension of the Sierpinski carpet?

本题中，我们假设以上极限是存在的。例如，谢尔宾斯基毯是下图的无穷迭代（下图中之显示了迭代的前两次）。通过无穷迭代，下图产生了自相似结构。这样的复杂物体叫做分形。每次迭代中，九个正方形里，正中的正方形空缺。谢尔宾斯基毯的计盒维数是多少呢？



Here, after each iteration, we can use squares to cover the image, with side length $s$ which is equal to the side length of the dark box. Let the length of the whole Sierpinski carpet be $L$. Then $B\left(s=\frac{L}{3}\right)=8, B\left(s=\frac{L}{9}\right)=64, \cdots, B\left(s=\frac{L}{3^n}\right)=8^n$. Note that here the limit $s \to 0$ is equivalent to the limit $n \to \infty$. Thus, the box counting dimension of the Sierpinski carpet is

这里，在每次迭代后，我们可以用边长 $s$ 等于深色方块的正方形来覆盖图形。假设整个谢尔宾斯基毯的长度为 $L$，则 $B\left(s=\frac{L}{3}\right)=8, B\left(s=\frac{L}{9}\right)=64, \cdots, B\left(s=\frac{L}{3^n}\right)=8^n$。注意到 $s \to 0$ 的极限就是 $n \to \infty$ 的极限，故谢尔宾斯基毯的计盒维数为

$$d_B=\lim_{s\to 0}\frac{\log B(s)}{\log(1/s)}=\lim_{n\to\infty}\frac{\log(8^n)}{\log(3^n/L)}=\frac{\log 8}{\log 3}.$$

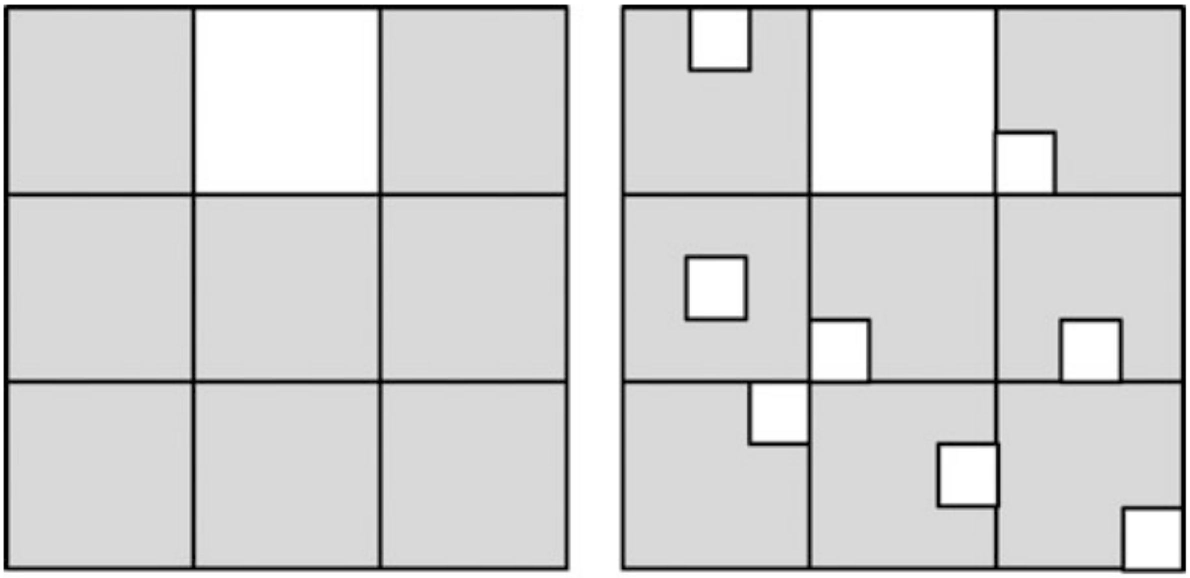
(Note that in the sense of taking limits, whichever unit we use, $\log L$ is negligible compared to $\log 3^n$ in the above equation.)

（注意到，在取极限的意义下，无论取什么单位，上式中 $\log L$ 和 $\log 3^n$ 相比都可以忽略。）

## A1. (2P) BOX COUNTING DIMENSION OF A RANDOMIZED SIERPINSKI CARPET 随机化谢尔宾斯基毯的计盒维数

The randomized Sierpinski carpet is the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations). In each iteration, a random square among the nine is removed. Calculate the box counting dimension of the randomized Sierpinski carpet.

随机化谢尔宾斯基毯是下图的无穷迭代（下图中只显示了迭代的前两次）。每次迭代中，九个正方形随机空缺一个。求随机化谢尔宾斯基毯的计盒维数。

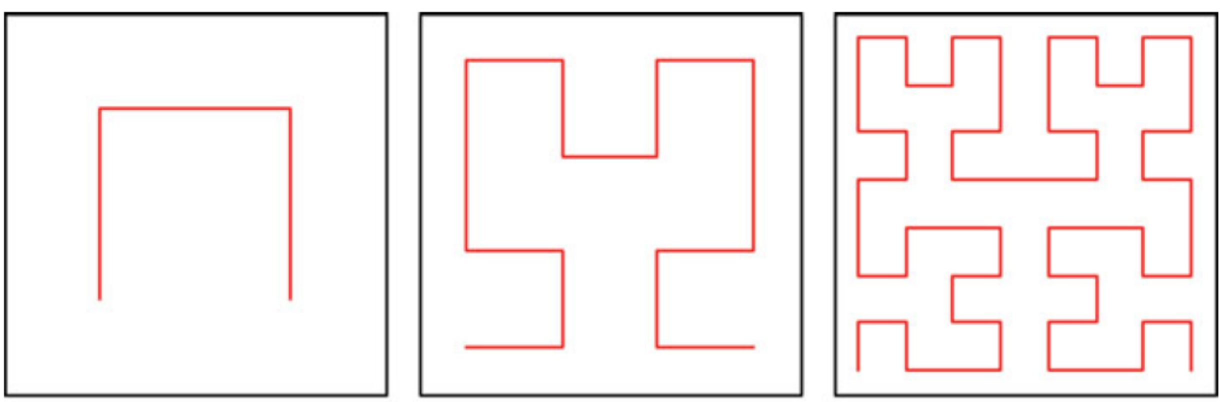


Solution: $\log 8/\log 3$ ($\simeq 1.893$, either the precise or approximate numerical number is fine). There is no difference between the above example (the position of the squire does not enter the calculation).

## A2. (2P) BOX COUNTING DIMENSION OF A HILBERT CURVE 希尔伯特曲线的计盒维数

The Hilbert curve is the infinite-time iteration of the following figure (in the following figure we only displayed the first three iterations). Calculate the box counting dimension of the Hilbert curve.

希尔伯特曲线是下图的无穷迭代（下图中只显示了迭代的前三次）。求希尔伯特曲线的计盒维数。



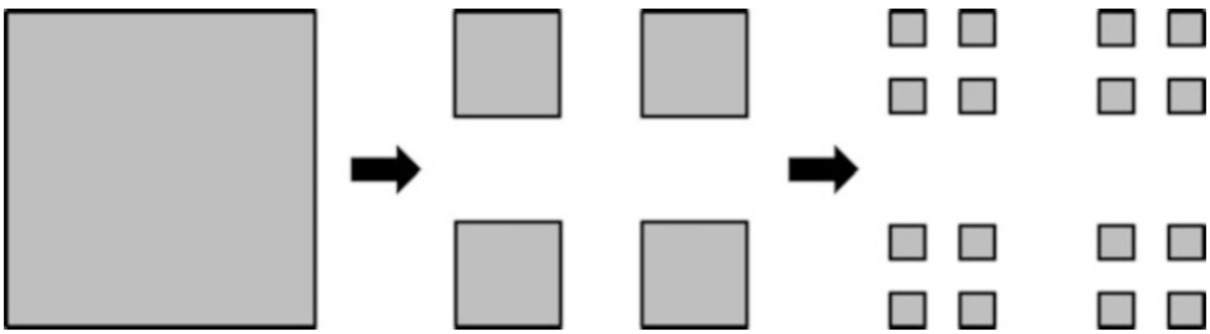
Solution: 2.

Because the Hilbert curve is space-filling. It passes through any given point. (This may not be easy to prove, but it's straightforward to see that for any given point, the curve will pass by close enough to this point.) Thus, no matter how small $s$ is, we always need to put boxes all over space to cover the whole curve.

## A3. (3P) BOX COUNTING DIMENSION OF THE CANTOR SET 康托尔集的计盒维数

The below is one form of Cantor set, defined as the infinite-time iteration of the following figure (in the following figure we only displayed the first two iterations)

一种形式的康托尔集是下图的无穷迭代（下图中只显示了迭代的前三次）。求康托尔集的计盒维数。

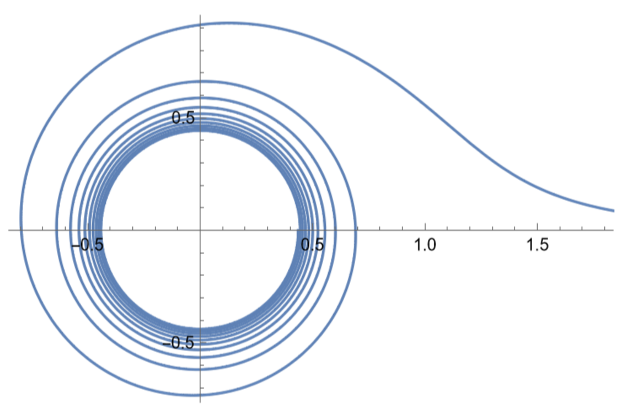


Solution: $B\left(s=\frac{L}{3^n}\right)=4^n$ (1p). Thus $d_B=\frac{\log 4}{\log 3}\simeq 1.262$ (2p)

## PART B. INTERSECTION BETWEEN A SPIRAL AND THE POSITIVE X-AXIS 螺旋线与正$x$轴的交点

Consider a spiral in the polar coordinate. Consider all the intersections between this spiral and the positive $x$-axis (i.e., the ray $\theta=0$ starting from the origin). We will call them “intersections” for short. What’s the box counting dimension of these intersections? We will solve this problem step-by-step in this part. (Note: in the calculation, we will allow $\theta\to+\infty$. In the below figure, we have not shown the large $\theta$ behavior.)

考虑极坐标下的螺旋线 $r(\theta)=\ \theta^{-\alpha}\ (\alpha>0)$。考虑该螺旋线与 $x$ 轴正方向（即从原点出发的 $\theta=0$ 射线）的所有交点（本部分 B 中，简称其为 “交点” ）。这些交点的计盒维数是多少？我们将分几个小题来解决这个问题。（注：在计算中我们将允许 $\theta\to+\infty$，下图中并没有显示出 $\theta$ 取很大值时的行为。）




## B1. (1P) THE VALUE OF $\theta$ OF THE INTERSECTIONS 交点的 $\theta$ 值

We denote the $\theta$ value of all the intersections by $\theta = \lambda j$, where $\lambda$ is a constant, $j$ is non-negative consecutive integers. Calculate $\lambda$ and calculate the range of $j$.

所有交点的 $\theta$ 值可以记作 $\theta = \lambda j$，其中 $\lambda$ 为常数，$j$ 为可连续取值的非负整数。求 $\lambda$ 的值，以及 $j$ 的取值范围。

Solution: $j = 1,2,\cdots$ (i.e. $j \neq 0$), $\lambda = 2\pi$, $\theta = 2\pi j$.

## B2. (1P) THE HORIZONTAL AXIS OF THE INTERSECTIONS 交点的横坐标

Calculate the horizontal axis $x_j$ of the intersection labelled by $j$, as a function of $j$ and $\alpha$.

求交点 $j$ 的横坐标 $x_j$，用 $j$ 和 $\alpha$ 表示。

Solution: $x_j = r\cos\theta = r = (2\pi j)^{-\alpha}$.

## B3. (3P) THE DISTANCE BETWEEN NEIGHBOR INTERSECTIONS 相邻交点的间距

Given a very small length $s$. Suppose $J$ is the smallest number which satisfying the following condition.

给定一个足够小的长度 $s$， 设 $J$ 满足如下条件的最小的数字：

$$x_j - x_{j+1} \leq s \text{ for all (对于所有) } j \geq J$$

Calculate $J$, in terms of $\alpha$ and $s$.

求 $J$ 的值，用 $\alpha$ 和 $s$ 表示。

Solution:

Since $s$ is sufficiently small, $J$ must be sufficiently large. (0.5p)

By definition, $J$ satisfies $(2\pi J)^{-\alpha} - [2\pi(J+1)]^{-\alpha} \leq s$ (1p)

Divide $(2\pi J)^{-\alpha}$ and Taylor expand the expression, we have $J \geq \frac{1}{2\pi}\left(\frac{2\pi\alpha}{s}\right)^{\frac{1}{\alpha+1}}$ (1p)

Since $J$ is the smallest number satisfying this inequality, $J = \frac{1}{2\pi}\left(\frac{2\pi\alpha}{s}\right)^{\frac{1}{\alpha+1}} = \left(\frac{\alpha}{(2\pi)^\alpha s}\right)^{\frac{1}{\alpha+1}}$ (0.5p)

## B4. (3P) BOX COUNTING 计盒数量

For small enough $s$, calculate $B(s)$ in terms of $\alpha$ and $s$. Note that here we use one-dimensional interval with length no more than $s$ (instead of two-dimensional boxes as given in Part A) as “boxes” to cover the intersections.

对足够小的 $s$，求 $B(s)$，用 $\alpha$ 和 $s$ 表示。注意，这里我们用长度至多为 $s$ 的一维线段（而不是 A 部分中的二维正方形）作为 “盒子” 来覆盖这些交点。

For $1 \le j \le J$, each of the intersection point needs to be covered by one box, since these points are far apart. For $j > J$, we need $x_J/s$ boxes to cover the entire $0 < x < x_J$ range. Thus $B(s) = J + \frac{x_J}{s}$ (2p, one for each term)

Note: since $J$ is large, answers such as $B(s) = (J-1) + \frac{x_J}{s}$ or $B(s) = J + \frac{x_{J+1}}{s}$ can be considered equally right and get full marks.

Insert the expressions for $J$ and $x_J$, we have

$B(s) = \left[\frac{1}{2\pi}(2\pi\alpha)^{\frac{1}{\alpha+1}} + (2\pi\alpha)^{-\frac{\alpha}{\alpha+1}}\right] s^{-\frac{1}{\alpha+1}}$ (1p, where coefficient 0.5p and power dependence of s 0.5p)

### B5. (1P) 计盒维数 BOX COUNTING DIMENSION

Calculate the box counting dimension of the intersections $d_B$ in terms of $\alpha$ and $s$.

求交点的计盒维数 $d_B$，用 $\alpha$ 和 $s$ 表示。

Take log of B4 in the small s limit, we have $d_B = 1/(1+\alpha)$.

## PART C 复杂网络的计盒维数 THE BOX COUNTING DIMENSION OF A COMPLEX NETWORK

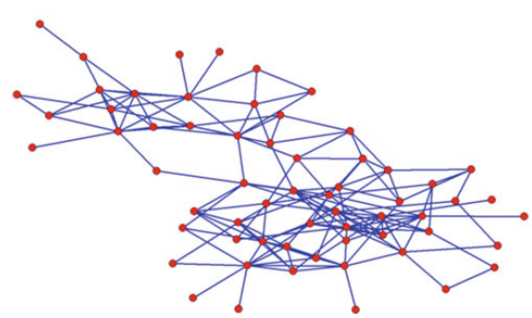
In the following, we will discuss network formed by vertices and their connections. We can use a generalized version of box counting dimension to analyze the feature of the network. Now the generalized "box" is no-longer square in the geometrical sense, but rather we require that the distance of any two vertices within a box to be no more than $s$ (i.e., a vertex in a box needs to travel at most $s$ edges to connect to any other given vertex). All the boxes can cover all the vertices. A vertex can only be in one box. The box counting dimension is calculated from the minimal box required.

下面，我们将讨论节点和节点之间的连接组成的网络。我们可以用推广的计盒维数来分析网络的特征。这时，推广的"盒子"不再是几何意义上的正方形，而是要求盒子里每两个节点间的距离不多于 $s$（也就是说一个节点最多通过 $s$ 条边就可以连接到盒子里的任一其它节点）。所有的盒子能覆盖所有节点，不同的盒子里包含的节点不能有交集。计盒维数可从满足以上条件的最少盒子数来计算。

### C1 (2P) 海豚的社交网络 THE SOCIAL NETWORK OF DOLPHINS

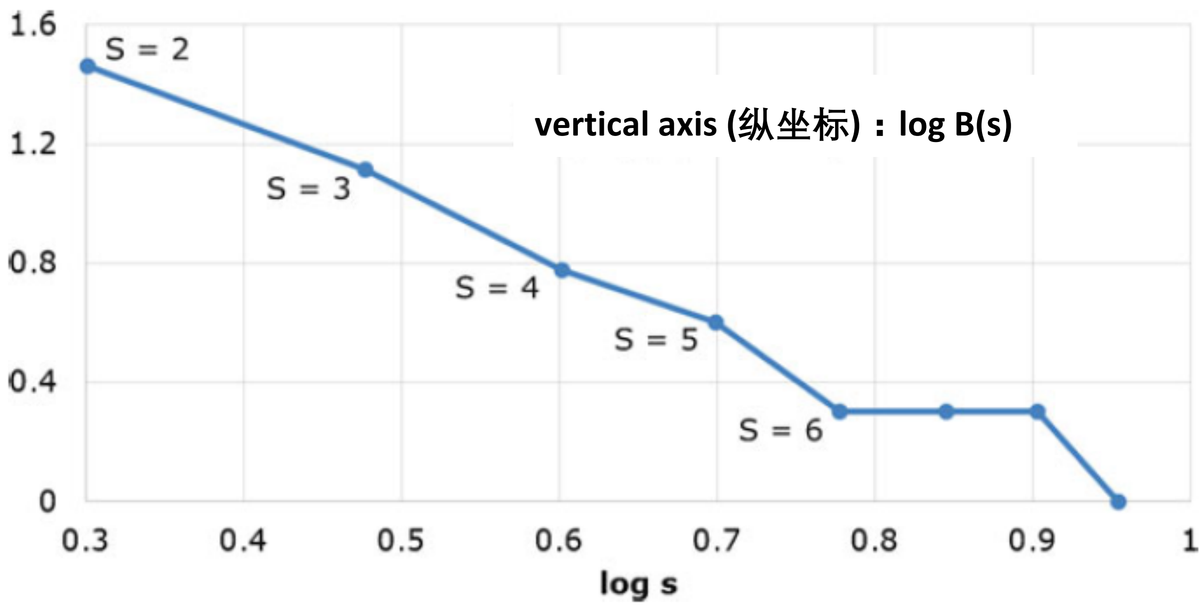
The below figure is a dolphin social network recorded by researchers:

下图为研究人员记录一个海豚群组的社交网络：



After analyze, we get

经过分析上图，得到




Suppose the slope of the plot at $s \to 0$ (in a complex network, since $s \geq 1$, the $s \to 0$ limit can only be understood in a sense of continuation) shows the same trend as the slope formed by the $s = 3$ and the $s = 2$ data points (for a fractal complex network, this slope should be a constant). Estimate the box counting dimension of the Dolphin network using the $s = 3$ and the $s = 2$ data points.

假设 $s \to 0$ 的曲线斜率（在复杂网络中，由于$s \geq 1$，$s \to 0$ 的极限只能在数学延拓的意义上定义）与 $s = 3$ 和 $s = 2$ 两个数据点连线的斜率体现出来的趋势相同（对于分形复杂网络，这个斜率应该是个常数），请用 $s = 3$ 和 $s = 2$ 两个数据点估计海豚网络的计盒维数。

Solution: We note that there is an unknown constant term in the box counting dimension. This constant term can be eliminated by taking the slope of the curve.

As we have assumed that the $s \to 0$ limit exists for calculating $d_B$, we have

$-d_B = \lim_{s\to 0} \frac{\log B(s)}{\log s} = \lim_{s\to 0} \lim_{\Delta s\to 0} \frac{\log B(s+\Delta s)}{\log(s+\Delta s)} = \lim_{s\to 0} \lim_{\Delta s\to 0} \frac{\log B(s+\Delta s)-\log B(s)}{\log(s+\Delta s)-\log s}$ (Proportional theorem)

From our assumption, we can use the slope of the $s = 3$ and $s = 2$ data points to estimate this slope. Thus,

$$d_B \simeq -\frac{\log B(3) - \log B(2)}{\log 3 - \log 2} \simeq 2.0$$

Answers between 1.5 ~ 2.5 are acceptable, which accounts for numerical errors when extracting data from figure.

## C2 (2P) 分形网络的尺度 THE SCALE OF THE FRACTAL NETWORK

Assuming the complex network has number of vertices $N_0 \gg 1$ with a fractal structure and box counting dimension $d_B$. Estimate the average distance $\bar{r}$ between nodes using $N_0$ and $d_B$ (as an estimate, we ignore the difference between average distance and maximal distance, and ignore $O(1)$ constants in the limit of an infinite network).

设复杂网络的节点个数为 $N_0 \gg 1$，网络具有分形结构，计盒维数为 $d_B$，请用 $N_0$ 和 $d_B$ 估计节点之间的平均距离 $\bar{r}$（作为估计，我们忽略平均距离和最大距离的区别，也忽略在网络无穷大极限下的 $O(1)$ 常数）。

Solution: From the experience of the previous question, we noted that $d_B$ can be estimated from the slope of the $(\log(s), \log B(s))$ curve. Take $s \sim O(1)$, from the fractal structure of the network, the nodes in the box $B(s)$ should behave similar to the whole network. Thus,

$$d_B \sim \frac{\log B(s) - \log B(s = \bar{r})}{\log \bar{r} - \log s} \sim \frac{\log B(s)}{\log \bar{r}}$$

Note that $\log B(s = \bar{r}) = O(1)$ and $\log s = O(1)$ are neglected in the last step of the above equation. Since we have taken $s \sim O(1)$, $B(s) \sim N_0$. Thus, $\bar{r} \sim N_0^{1/d_B}$.

Alternative Solution: The students may get an intuitive solution $\bar{r} \sim N_0^{1/d_B}$ directly, by imaging $\bar{r}$ and $N_0$ as the length and volume of a $d_B$-dimensional cube. Though mathematically this is not rigorously following our definitions, we can also give full marks to this solution.

## PART D 复杂网络的重整化 THE RENORMALIZATION OF COMPLEX NETWORKS

When calculating the box counting dimension, we use boxes to contain vertices. Now we can construct a new complex network: the vertices of the new network corresponds to the boxes of the original network. If in the original network, a vertex in a box connects to at least one vertex in another box, then the two boxes of the new network (considered as two vertices in the new network) is connected. For example,

在计算计盒维数的时候，我们用盒子把节点“装”起来。这时，我们可以构造一个新的复杂网络：新网络的节点是原复杂网络的盒子；如果旧网络一个盒子里的一个节点至少连接到另一个盒子里的任何节点，则新网络里这两个盒子（即新网络里的两个节点）相连。例如：


### D1 (2P) 重整化后的计盒维数 THE BOX COUNTING DIMENSION AFTER RENORMALIZATION

For a fractal network, let the box counting dimension of the original network be $d_B$, after renormalization with box size $s$, calculate the box counting dimension of the new network.

对分形网络，设原网络的计盒维数为 $d_B$，在进行盒子尺度为 $s$ 的重整化后，求新网络的计盒维数。

Solution: $d_B$ (because of self-similarity. Or one can calculate using slope, taking boxes with size larger than $s$.)

## D2 (4P) 向网络添加长程连接 ADDING LONG RANGE CONNECTIONS TO A NETWORK

Many complex networks in our real life do not look like a fractal network. For example, you may have heard the "six degrees of separation", that through at most six people, you can use "friend of friend of friend of friend of friend of friend" connection to know any person in the world. Thus, we often feel "what a small world"!

我们现实世界中的很多复杂网络看上去并不像分形网络。例如，你可能听说过"六度分隔理论"，就是说，最多通过六个人，你能以"朋友的朋友的朋友的朋友的朋友的朋友"的方式认识世界上任何一个人。因此，我们经常惊呼"世界太小了"。

Usually in a fractal network there are too few long-range connections, not enough to have the "small world" feature. To describe a "small world", we randomly add long-range connections to a network: For any pair of vertices with distance $r$ ($r > 1$), we randomly add connections to these vertices with probability $p(r) = Ar^{-\alpha}$ $(\alpha > 0)$. For large enough $r$, these newly added connections dominate the connections of the new network.

通常的分形网络中这样的长程连接非常少，不足以描述"小世界"。为了描述这样的"小世界"现象，我们向一个分形网络中随机添加一些长程连接：对于任何距离为 $r$ 的两个节点（$r > 1$），我们以概率 $p(r) = A\,r^{-\alpha}$ $(\alpha > 0)$ 的概率连接这两个节点。对于足够大的 $r$，新添加的这些连接是新网络的主要连接方式。

Now, we perform renormalization of this network with box size $s$. After renormalization, the new distance $r_s$ of the new network is related to the original distance $r$ of the original network by $r_s = r/s$. After renormalization, for the new network, at sufficiently large distances, calculate the probability $p_s(r_s)$ that two vertices are connected, as a function of $A, s, \alpha, d_B, r_s$. (Here $d_B$ is the fractal dimension of the original network before adding long range connection. Before adding long range connections, we can consider the network in the box as a sub-network similar to the whole network.)

现在，我们对网络做盒子尺度为 $s$ 的重整化。重整化后，新网络上的距离 $r_s$ 与旧网络上的距离 $r$ 的关系为 $r_s = r/s$。求重整化后，新网络在足够大距离上，两个节点之间的连接概率 $p_s(r_s)$，用 $A, s, \alpha, d_B, r_s$ 表示。（这里 $d_B$ 为添加长程链接前，原网络的计盒维数。在添加长程连接前，可将盒子里的网络看成是一个与整个网络相似的子网络。）

Solution:

Consider the old network. Let the number of nodes in a typical renormalization box be $N_B$. Since $s \gg 1$, we have $N_B = s^{d_B}$. (1p)

For two such boxes, the probability of no connection between two boxes is $[1 - p(r)]^{N_B^2}$ (2p). Note here 2 in $N_B^2$ is because you can first calculate the probability between one box and any single node in the other box. This factor of 2 deserves 1 point.

Thus, $p_s(r_s) = 1 - (1 - A(sr_s)^{-\alpha})^{s^{2d_B}}$

## D3 (4P) 重整化群的不动点 THE FIXED POINT OF A RENORMALIZATION GROUP

For a large enough network $G$, we can apply the renormalization procedure $R$ repeatedly, to get $R(G)$, $R(R(G))$, $R(R(\ldots R(G)\ldots))$ $R(G), R(R(G)), R(R(\ldots R(G)\ldots))$. This repeated renormalization procedure is known as the renormalization group. If after sufficiently many operations, the statistical property of the network no longer changes (in this problem the probably of long-range connections no longer changes), we call the network after many renormalization the "fixed point" of renormalization group (here, the "point" in fixed point means that, in the space of all networks, each network is considered as a point). Since infinite iteration is difficult technically, we alternatively take one renormalization with the $s \to \infty$ (but still $s \ll \bar{r}$) limit. For different values of $\alpha$, after adding long-range connections with probability $p(r) = A\, r^{-\alpha}$ $(\alpha > 0)$, calculate the expression of $p(r)$ on the fixed point (i.e., determine all possible cases of $p_s(r_s)$ as $r_s \to \infty$). You may use an identity $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

对于足够大的网络 $G$，我们可以反复进行重整化操作 $R$，得到 $R(G), R(R(G)), R(R(\ldots R(G)\ldots))$。这种反复的重整化操作叫做重整化群。如果进行足够多次重整化操作后，网络的统计性质不再改变（本题中体现为长程连接的概率不再改变），我们称这样做过很多次重整化的网络为重整化群的"不动点"。（这里点的意思是，所有网络组成的空间中，每个网络看成是其中一个点）。由于无穷次迭代在技术上较困难，我们也可以取单次 $s \to \infty$ (但是仍然满足 $s \ll \bar{r}$) 的极限来代替无穷次迭代的操作。根据 $\alpha$ 的不同取值，求以概率 $p(r) = A\, r^{-\alpha}$ $(\alpha > 0)$ 添加长程连接之后，复杂网络的不动点上 $p(r)$ 的表达式 (即在 $r_s \to \infty$ 的极限下计算所有 $p_s(r_s)$ 的极限情况)。你可能会用到 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$。

Solution:

Note that $(1 - A(sr_s)^{-\alpha})^{s^{2d_B}} = [(1 - A(sr_s)^{-\alpha})^{s^\alpha}]\wedge(s^{2d_B-\alpha}) \simeq \exp\left(-A\, r_s^{-\alpha} s^{2d_B-\alpha}\right)$ . (1p)

Thus, $p_s \simeq 1 - \exp\left(-A\, r_s^{-\alpha} s^{2d_B-\alpha}\right)$ has three cases:

When $2d_B - \alpha > 0$, $p_s \to 1$ (complete graph fixed point). (1p)

When $2d_B - \alpha < 0$, $p_s \to 0$ (fractal fixed point). (1p)

When $2d_B - \alpha = 0$, $p_s \to 1 - \exp(-A\, r^{-2d_B})$ (small-world fixed point). (1p)

Note: the names of the fixed points just indicate the physical meaning. The students only need to write down the different cases and probabilities (each 0.5p), no need to write the names of the fixed points.

References:

[1] "Fractal Dimensions of Networks", by Eric Rosenberg, Springer Press.

[2] "Small-World to Fractal Transition in Complex Networks: A Renormalization Group Approach", Rozenfeld, Song and Makse, Physical Review Letters 104, 025701 (2010).