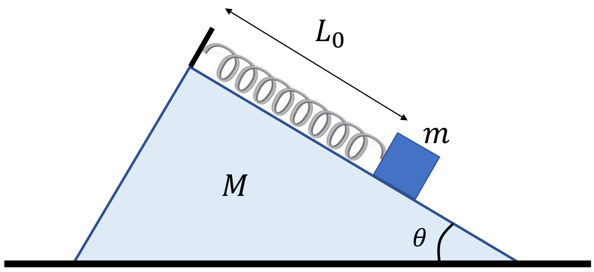
Pan Pearl River Delta Physics Olympiad 2022
2022 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST
香港科技大学高等研究院赞助
Simplified Chinese Part-1 (Total 4 Problems, 40 Points) 简体版卷-1（共4题，40分）
(9:30 am – 12:00 pm, 7th February 2022)

Please fill in your final answers to all problems on the answer sheet.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

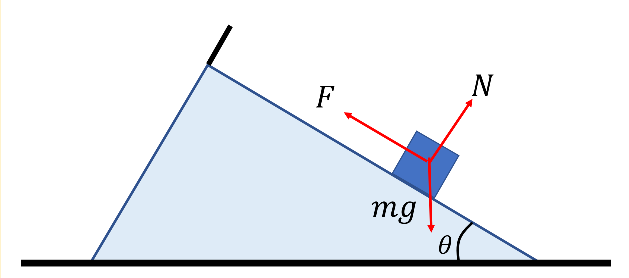
1. [10 points] A wooden wedge of mass M is placed on a smooth table. A small wooden block of mass m is connected to the top of the wooden wedge with a spring of elastic constant k. Suppose the natural length of the spring is L_0, and there is no friction between two wooden blocks. Now the small block is released at rest at distance L_0 from the top of the wooden wedge (as shown in the picture) and slides down freely. Find
(a) [1pt] The equilibrium position of the small block m measured from the top of the wooden wedge.
(b) [2pt] The horizontal distance travelled by the wooden wedge when the small block m reaches its equilibrium position, i.e. the position where the acceleration of m vanishes.
(c) [1pt] The oscillating amplitude of the small block m along the slope of the wedge.
(d) [6pt] The period of oscillation of the small block m along the slope of the wedge.

1. [10 分] 一塊楔形木塊，質量為 M ，被置放於一光滑之桌面上，另一質量為 m 之小木塊以一彈力常數為 k 的彈簧繫於楔形木塊之頂端。設彈簧之原長為 L_0，且兩木塊之間無摩擦。今將小木塊自離木塊頂端 L_0 處靜止釋放（如圖所示），讓其自由滑下。試求



(a) [1 分] 小木块 m 之平衡位置离楔形木块顶端之距离。
(b) [2 分] 小木块 m 到达其平衡位置时（m 的加速度为零的位置），楔形木块移动之水平距离。
(c) [1 分] 小木块 m 沿楔形木块斜面之振幅。
(d) [6 分] 小木块 m 来回振荡之周期。

Solution:
(a) There are 3 forces acting on the block m.



At equilibrium,

$$F = mg\sin\theta = k(l_{eq} - l_0)$$
$$\Rightarrow l_{eq} = \frac{mg\sin\theta}{k} + l_0$$

(b) Let the horizontal position of wedge and small block be X and x respectively. Initially, $x = X = 0$. At equilibrium position, we have

$$mx + MX = 0$$

From part (a),

$$x - X = (l - l_0)\cos\theta = \frac{mg}{k}\sin\theta\cos\theta$$

Solve 2 equations, we get

$$X = -\frac{mg\sin\theta}{k}\frac{m}{m+M}\cos\theta$$

The horizontal distance travelled by the wedge is $\frac{mg\sin\theta}{k}\frac{m}{m+M}\cos\theta$.

(c) (Method 1) The maximum distance travelled by the block m is the distance between the initial and the equilibrium position,

$$A = |l_0 - l_{eq}| = \frac{mg}{k}\sin\theta$$

(Method 2)
Assume the maximum distance travelled by the block m is l_m. From the energy conservation,

$$mg(l_m - l_0)\sin\theta = \frac{1}{2}k(l_m - l_0)^2$$
$$\Rightarrow l_m - l_0 = \frac{2mg}{k}\sin\theta$$

And the amplitude is given by the maximum distance from the equilibrium position,

$$A = l_m - l_{eq} = \frac{mg}{k}\sin\theta$$

(d) Let the vertical and horizontal acceleration of mass m be a_y and a_x and the horizontal acceleration of the wedge M is A_x. We have

$$ma_x + MA_x = 0. \quad (1)$$
$$\frac{a_x - A_x}{a_y} = \cot\theta \quad (2)$$
$$-N\sin\theta + F\cos\theta = MA_x \quad (3)$$
$$mg - N\cos\theta - F\sin\theta = ma_y \quad (4)$$
$$F = k(l - l_0) = \frac{k}{\cos\theta}(x - X) = \frac{kx}{\cos\theta}\left(1 + \frac{m}{M}\right) \quad (5)$$

From eqtn (1) and (2),

$$a_y = (a_x - A_x)\tan\theta = a_x\left(1 + \frac{m}{M}\right)\tan\theta$$

Substitute into (4),

$$mg - N\cos\theta - F\sin\theta = ma_x\left(1 + \frac{m}{M}\right)\tan\theta \quad (6)$$

From (3) and (6), we can eliminate N and

$$mg\sin\theta - F = ma_x\left(1 + \frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} - MA_x\cos\theta = ma_x\left[\left(1 + \frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right]$$

Substitute F from eqtn (5),

$$mg\sin\theta - \frac{kx}{\cos\theta}\left(1 + \frac{m}{M}\right) = ma_x\left[\left(1 + \frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right]$$

We can find a term $a_x \propto -x$ in the expression which gives rise the SHM. The angular frequency is

$$\omega^2 = \frac{\frac{k}{\cos\theta}\left(1 + \frac{m}{M}\right)}{m\left[\left(1 + \frac{m}{M}\right)\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right]} = \frac{k\left(1 + \frac{m}{M}\right)}{m[\left(1 + \frac{m}{M}\right)\sin^2\theta + \cos^2\theta]}$$

$$\Rightarrow \omega^2 = \frac{k(M+m)}{m[M + m\sin^2\theta]}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m(M + m\sin^2\theta)}{k(M+m)}}$$

We can check the result in several limits:

(1) As $M \to \infty, \omega^2 \to \frac{k}{m}$

(2) As $\theta \to 0$,

$$\omega^2 \to \frac{k}{m}\left(1 + \frac{m}{M}\right) = \frac{k}{\left(\frac{mM}{m+M}\right)}$$

(3) As $\theta \to \frac{\pi}{2}$,

$$\omega^2 = \frac{k}{m}$$

(Method 2) Consider the motion of the wedge M. Fundamentally, the motion of M is complicated: it is pulled by the spring and at the same time also pressed by m. But effectively, since it is under harmonic motion, we can consider the spring and m together as an effective spring with spring constant k_{eff}. The angular frequency is $\omega^2 = k_{\text{eff}}/M$, and from energy conservation,

$\frac{1}{2}k_{\text{eff}}X^2 = \frac{1}{2}M\,V^2$ (7), where V is the velocity of the wedge at the equilibrium position.

Here V can be solved by energy conservation and momentum conservation of the system. Let the velocity of m with respect to M be $\vec{v}_{\text{r}}$, then $\vec{v}_{\text{r}}$ satisfies $v_{\text{r}y} = v_{\text{r}x}\tan\theta$.

Let the velocity of m with respect to the ground be $\vec{v}$. Then, $v_x = v_{\text{r}x} - V$, $v_y = v_{\text{r}y}$.

Momentum conservation of the x-direction: $MV = mv_x \quad \Rightarrow \quad v_x = \frac{M}{m}V$, $v_y = \frac{M+m}{m}V\tan\theta$.

Energy conservation: $\frac{1}{2}mg \times \frac{mg\sin^2\theta}{k} = \frac{1}{2}m\left(v_x^2 + v_y^2\right) + \frac{1}{2}MV^2$.

Inserting v_x and v_y into the above equation, solve $\frac{1}{2}MV^2$ and then use eqtn (7) to solve k_{eff}, we have

1

$k_{\text{eff}} = \frac{kM(M+m)}{m(M+m\sin^2\theta)}$. Thus, the frequency is $\omega^2 = \frac{k_{\text{eff}}}{M} = \frac{k(M+m)}{m[M+m\sin^2\theta]}$.

2. [10 points] A system of 3 energy levels, $E_1 = 0, E_2 = \epsilon$, and $E_3 = 10\epsilon$ ($\epsilon > 0$) is populated by $N \gg 1$ particles at temperature T. The particles populate the energy levels according to the classical Boltzmann distribution law.
(a) [2pt] What is the average number of particles, N_3, with energy E_3?
(b) [2pt] What is the average energy of a particle at temperature T?
(c) [2pt] At sufficiently low temperature T_c, only energy levels E_1, E_2 are populated. Calculate the order of magnitude of the characteristic temperature T_c.
(d) [2pt] Calculate the molar specific heat at constant volume C_v at low temperature $T \ll \epsilon$.
(e) [2pt] Calculate the molar specific heat at constant volume C_v at high temperature $T \gg \epsilon$.

2. [10 分] 一个由 3 个能级 $E_1 = 0$、$E_2 = \epsilon$ 和 $E_3 = 10\epsilon$ ($\epsilon > 0$) 组成的系统在温度 T 下由 $N \gg 1$ 个粒子填充。这些粒子根据经典 Boltzmann 分布定律填充能级。

(a) [2 分] 具有能量 E_3 的粒子的平均数量 N_3 是多少？
(b) [2 分] 粒子在温度 T 下的平均能量是多少？
(c) [2 分] 在足够低的温度 T_c 下，仅填充 E_1、E_2 兩个能级。计算特征温度 T_c 的数量级。
(d) [2 分] 计算低温 $T \ll \epsilon$ 下等体积摩尔比热 C_v。
(e) [2 分] 计算高温 $T \gg \epsilon$ 下等体积摩尔比热 C_v。

Solution:
(a) We have

$$N_1 + N_2 + N_3 = N$$
$$\frac{N_2}{N_1} = e^{-\frac{\epsilon}{kT}}$$
$$\frac{N_3}{N_1} = e^{-\frac{10\epsilon}{kT}}$$
$$\Rightarrow N_1 = \frac{N}{1 + e^{-\frac{\epsilon}{kT}} + e^{-\frac{10\epsilon}{kT}}}$$

And

$$N_3 = \frac{Ne^{-\frac{10\epsilon}{kT}}}{1 + e^{-\frac{\epsilon}{kT}} + e^{-\frac{10\epsilon}{kT}}}$$

(b) The average energy of a particle is

$$E = \frac{\epsilon e^{-\frac{\epsilon}{kT}} + 10\epsilon e^{-\frac{10\epsilon}{kT}}}{1 + e^{-\frac{\epsilon}{kT}} + e^{-\frac{10\epsilon}{kT}}}$$

(c) If E_3 is unpopulated, we have $N_3 < 1$. At the characteristic temperature T_c,

$$N_3 = 1$$
$$\Rightarrow Ne^{-\frac{10\epsilon}{kT_c}} = 1 + e^{-\frac{\epsilon}{kT_c}} + e^{-\frac{10\epsilon}{kT_c}}$$
$$\Rightarrow N = e^{\frac{10\epsilon}{kT_c}} + e^{\frac{9\epsilon}{kT_c}} + 1$$

If $N \gg 1$,

$$\Rightarrow N \approx e^{\frac{10\epsilon}{kT_c}}$$
$$\Rightarrow T_c \approx \frac{10\epsilon}{k \ln N} = O\left(\frac{\epsilon}{k \ln N}\right)$$

(d) The molar specific heat is given by

$$C_V = N_A \frac{\partial E}{\partial T} = (kN_A\epsilon^2\beta^2)\frac{\left(e^{-\beta\epsilon} + 100e^{-10\beta\epsilon} + 81e^{-11\beta\epsilon}\right)}{(1 + e^{-\beta\epsilon} + e^{-10\beta\epsilon})^2}$$

At low temperature $T \ll \epsilon$,

$$C_V \approx \left(\frac{N_A\epsilon^2}{kT^2}\right)e^{-\frac{\epsilon}{kT}}$$

(e) At high temperature $T \gg \epsilon$ (i.e. $\beta\epsilon \ll 1$),

$$C_V \approx (kN_A\epsilon^2\beta^2)\frac{(1 + 100 + 81)}{(3)^2} = \frac{182}{9}\frac{N_A\epsilon^2}{kT^2}$$

3. [10 points] One cylindrical vessel of radius R_1 is fixed inside another cylindrical vessel of radius R_2, as shown in the figure. In the bottom of the small vessel, there is a small hole with a bushing and a wooden cylinder of radius r and height $h = 21\text{cm}$ is inserted. The wooden cylinder can only move vertically relative to bushing without friction. Water is poured into the small vessel to a height of $a = 30\text{cm}$, and oil is poured into the large vessel to the same level. And the wooden cylinder is in equilibrium.

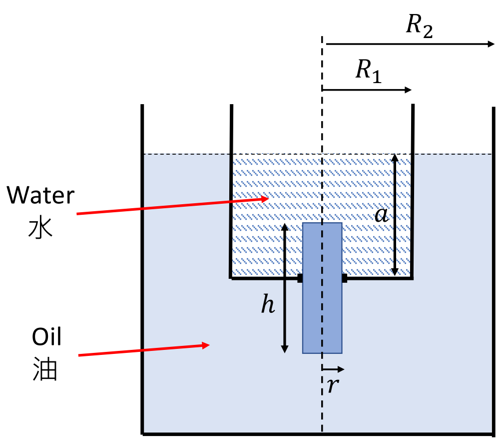
Given that the water density is $\rho_w = 1000\text{kg/m}^3$, the oil density is $\rho_o = 790\text{kg/m}^3$ and the wooden cylinder density is $\rho = 600\text{kg/m}^3$.

(a) [5 pt] Find the fraction of the wooden cylinder is in the water?
(b) [5 pt] Find the condition between ρ_w, ρ_o, r, R_1 and R_2 such that the equilibrium of the wooden cylinder is stable. (Hint: You need to consider the finite size effect of R_1, R_2 and r)

3. [10 分] 一个半径为 R_1 的圆柱容器固定在另一个半径为R_2 的圆柱容器内，如图所示。在小容器的底部有一个带衬套的小孔，插入半径为 r、高为 $h = 21\text{cm}$ 的木圆柱。木圆柱只能相对于衬套垂直移动而无摩擦。将水倒入小容器至 $a = 30\text{cm}$ 的高度，将油倒入大容器至同一高度。并且木圆柱处于平衡状态。

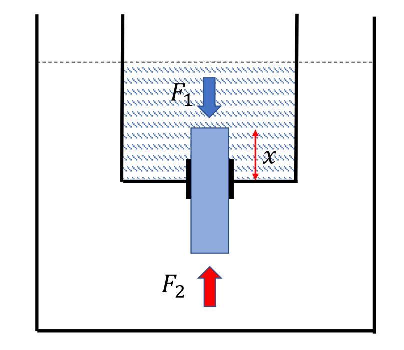
假设水的密度为 $\rho_w = 1000\text{kg/m}^3$，油的密度为 $\rho_o = 790\text{kg/m}^3$，木圆柱的密度为 $\rho = 600\text{kg/m}^3$。

(a) [5 分] 木圓柱在水中的部分與全長的比例?
(b) [5 分] 找出 ρ_w, ρ_o, r, R_1 和 R_2 之间的条件，使得木圆柱的平衡是稳定的。（提示：您需要考虑 R_1、R_2 和 r 的有限尺寸效应）



Solution:

(a) At equilibrium, we have



$$P_1 + \rho g h = P_2$$

$$P_1 = P_{atm} + \rho_w g(a - x)$$

$$P_2 = P_{atm} + \rho_o g(a + h - x)$$

$$\Rightarrow \rho_w(a - x) + \rho h = \rho_0(a + h - x)$$

$$\Rightarrow (\rho_o - \rho_w)x = -\rho h + \rho_0(a + h) - \rho_w a$$

$$\Rightarrow x = \frac{(\rho - \rho_0)h + (\rho_w - \rho_o)a}{\rho_w - \rho_o}$$

$$\text{Fraction of the cylinder in the water} = \frac{x}{h} = \frac{(\rho - \rho_0) + (\rho_w - \rho_o)\frac{a}{h}}{\rho_w - \rho_o} = \frac{-190 + 210 \times \frac{30}{21}}{210}$$

$$= -\frac{19}{21} + \frac{30}{21} = \frac{11}{21}$$

(b) If $x \to x + \Delta x$ ($\Delta x > 0$), the water level will rise and the oil level will drop.

The rise of the water level is

$$\pi R_1^2(a + \Delta a_w) - \pi r^2(x + \Delta x) = \pi R_1^2 a - \pi r^2 x$$

$$\Rightarrow \Delta a_w = \frac{r^2}{R_1^2}\Delta x$$

Similarly,

$$\pi(R_2^2 - R_1^2)\Delta a_0 = \pi r^2 \Delta x \Rightarrow \Delta a_o = \frac{r^2}{R_2^2 - R_1^2}\Delta x$$

The net force acting on the cylinder is

$$F_{net} = (P_1 + \rho g h - P_2)\pi r^2$$

$$P_1 = P_{atm} + \rho_w g(a + \Delta a_w - x - \Delta x)$$

$$P_2 = P_{atm} + \rho_o(a - \Delta a_o + h - x - \Delta x)$$

$$\Rightarrow F_{net} = \rho_w g(a + \Delta a_w - x - \Delta x) + \rho g h - \rho_o g(a - \Delta a_0 + h - x - \Delta x)$$

$$= \rho_w g(\Delta a_w - \Delta x) + \rho_0 g(\Delta a_0 + \Delta x) = \rho_w g\left(\frac{r^2}{R_1^2} - 1\right)\Delta x + \rho_0 g\left(\frac{r^2}{R_2^2 - R_1^2} + 1\right)\Delta x$$

$$= \left(\rho_w\left(\frac{r^2}{R_1^2} - 1\right) + \rho_0\left(\frac{r^2}{R_2^2 - R_1^2} + 1\right)\right) g\Delta x$$

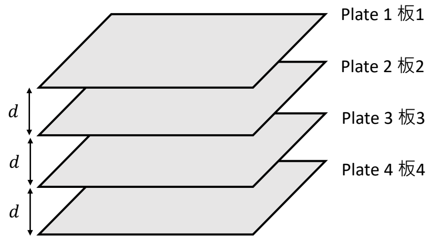
The force will push the cylinder into the equilibrium position if $F_{net} > 0$

$$\Rightarrow \rho_w\left(\frac{r^2}{R_1^2} - 1\right) + \rho_0\left(\frac{r^2}{R_2^2 - R_1^2} + 1\right) > 0$$

$$\Rightarrow \rho_w\left(\frac{R_1^2 - r^2}{R_1^2}\right) < \rho_0\left(\frac{R_2^2 - R_1^2 + r^2}{R_2^2 - R_1^2}\right)$$

4. [10 points] Four square metal plates of area A are arranged at an even spacing d as shown in the diagram. (Assume that $A \gg d^2$)

4. [10 分] 如图所示，四块面积为 A 的方形金属板以等间距 d 排列。（假设$A \gg d^2$）



We perform the following steps to the system:
Step 1: Plate 1 and 4 are first connected to a voltage source of magnitude V_0, with plate 1 positive; Plate2 and 3 are connected with a wire.
Step 2: Remove the voltage source between plate 1 and 4.
Step 3: Remove the wire between plate 2 and 3.
Step 4: Finally, plate 1 and 4 are connected by a wire.

我们对系统执行以下步骤：

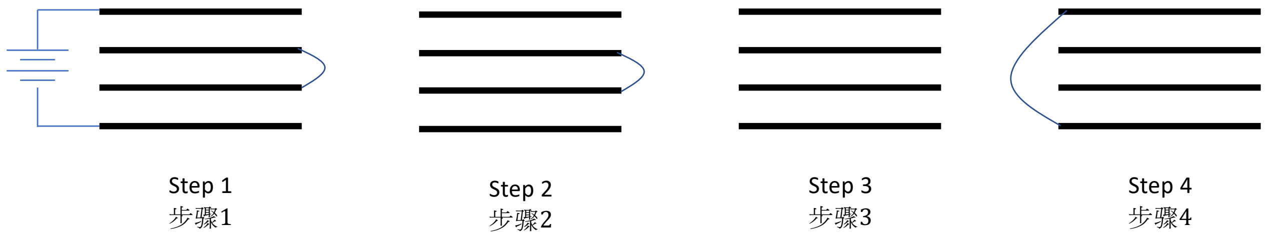
步骤 1：板 1 和 4 首先连接到幅度为V_0 的电压源，板 1 为正极；板 2 和板 3 用电线连接。

步骤 2：移除板 1 和板 4 之间的电压源。

步骤 3：拆下板 2 和 3 之间的电线。

步骤 4：最后，板 1 和 4 用电线连接。

The steps are summarized in the diagrams below.
下图总结了这些步骤。



(a) [6pt] Find the potential difference ΔV_{12}, ΔV_{23} and ΔV_{34} at step 4, where $\Delta V_{ij} = V_i - V_j$ is the potential difference between plate i and j.

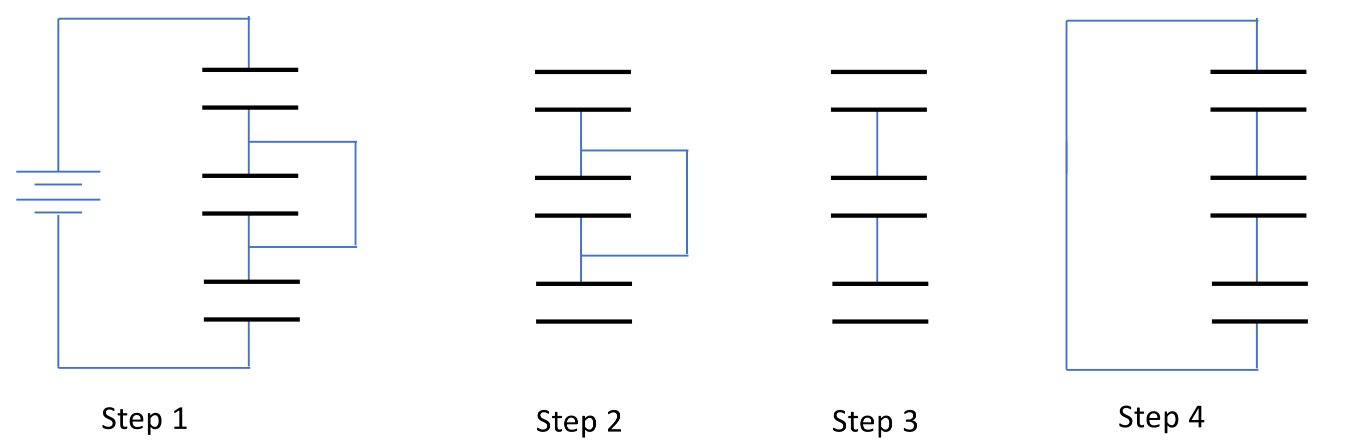
(a) [6 分] 求步骤 4 中的电位差 ΔV_{12}、ΔV_{23} 和 ΔV_{34}，其中 $\Delta V_{ij} = V_i - V_j$ 为板 i 和 j 之间的电位差。

(b) [4pt] What is the net electrostatic force acting on the plate 1 at step 4?

(b) [4 分] 求步骤 4 作用在板 1 上的净静电力是多少？

Solution:

We treat the plates as three capacitors in series. Each has an identical capacitance C. The figure below then show the 4 steps.



Since C_2 is shorted initially, effectively there are only two capacitors in series. The voltage drop across C_1 and C_3 is $V_0/2$. The top plate of C_1 will then have a positive charge of $q_0 = \frac{CV_0}{2}$. Note that this means that the bottom plate of C_1 will have a negative charge of $-q_0$. Removing the voltage source and then the wire across C_2 will not change the charges or potential drops across the other two capacitors.

Shorting the top plate of C_1 with the bottom plate of C_3 will make a difference. Positive charge will flow out of the top plate of C_1 into the bottom plate of C_3. Also, negative charge will flow out of the bottom plate of C_1 into top plate of C_2. The result is that C_1 will acquire a potential difference of V_1, C_2 a potential difference of V_2 and C_3 a potential difference of V_3.

Let the final charge on the top plate of each capacitor be q_1, q_2 and q_3 respectively.

Kirchhoff's loop rule implies,

$$V_1 + V_2 + V_3 = 0$$

By symmetry, we have

$$V_1 = V_3$$

$$\Rightarrow 2V_1 = -V_2$$

By charge conservation between the bottom plate of C_1 and the top plate of C_2, we have

$$-q_0 = -q_1 + q_2 \Rightarrow -\frac{1}{2}V_0 = -V_1 + V_2$$

$$\Rightarrow V_2 = -\frac{1}{3}V_0$$

And

$$V_1 = \frac{1}{6}V_0.$$

Final answer:

$$\Delta V_{12} = \frac{1}{6}V_0, \qquad \Delta V_{23} = -\frac{1}{3}V_0, \quad \Delta V_{34} = \frac{1}{6}V_0$$

(b) The charge on the top plate is

$$q_1 = C\Delta V_{12} = \frac{1}{6}\frac{\epsilon_0 A}{d}V_0$$

And the electric field inside the top capacitor is

$$E_1 = \frac{\Delta V_{12}}{d} = \frac{V_0}{6d}$$

The electric force acts on the top plate is

1

$$F = \frac{1}{2} E_1 \times q_1 = \frac{1}{2} \frac{V_0}{6d} \frac{\epsilon_0 A}{6d}\ V_0 = \frac{1}{72} \frac{\epsilon_0 A}{d^2} V_0^2$$