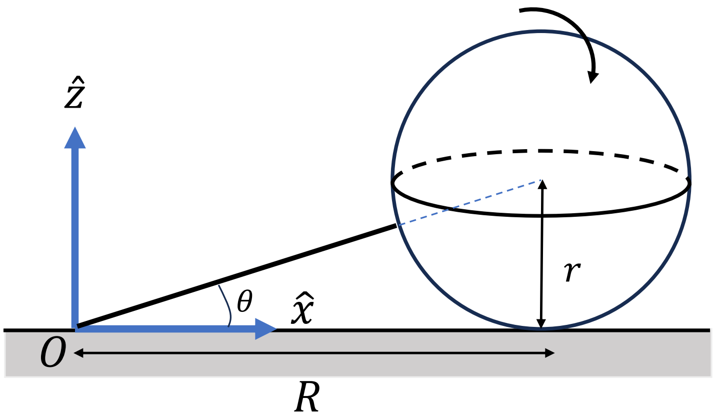
Pan Pearl River Delta Physics Olympiad 2025
2025 年泛珠三角及中华名校物理奥林匹克邀请赛
Sponsored by Institute for Advanced Study, HKUST. 香港科技大学高等研究院赞助
Paper-1 (Total 4 Problems, 10 Points each) 卷-1（共4题，每题10分）(9:30 am – 12:00 pm, 2nd Feb 2025)

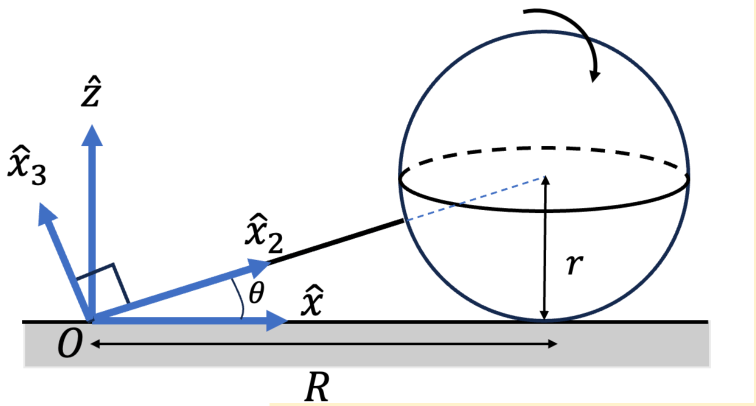
Please fill in your final answers to all problems on the answer sheet.
请在答题纸上填上各题的最后答案。
At the end of the competition, please submit the answer sheet only. Question papers and working sheets will not be collected.
比赛结束时，请只交回答题纸，题目纸和草稿纸将不会收回。

1. Consider a lollipop made of a solid sphere of mass m and radius r that is radially pierced by a massless stick. The free end of the stick is pivoted on the ground. The sphere rolls on the ground without slipping, with its centre moving in a circle of radius R with angular velocity Ω (along $-\hat{z}$ direction). The moment of inertia of a solid sphere along the symmetric axis with mass m and radius r is $I = \frac{2}{5}mr^2$.
(a) [3] What is the (instantaneous) total angular velocity $\vec{\omega}$ of the lollipop? Express your answer in terms of $r, R, \Omega, \hat{x}$ and $\hat{z}$.
(b) [4] What is the angular momentum $\vec{L}$ of the lollipop? Express your answer in terms of $m, r, R, \Omega, \hat{x}$ and $\hat{z}$.
(c) [3] What is the normal force $\vec{N}$ between the ground and the sphere?

1. 一枝棒棒糖可想像为一个质量为 m，半径为 r 的实心球被一根质量可忽略不计的棒沿半径方向贯穿。这根棒的另一端固定在地面作为支点。实心球在地面上以不打滑的方式滚动，其球心以半径为 R 的圆周运动，并以角速度 Ω（沿 $-\hat{z}$ 方向）旋转。已知质量为 m，半径为 r 实心球沿对称轴的转动惯量为 $I = \frac{2}{5}mr^2$。
(a) [3] 棒棒糖的瞬时总角速度 $\vec{\omega}$ 是多少？请用 $r, R, \Omega, \hat{x}, \hat{z}$ 表示你的答案。
(b) [4] 棒棒糖的角动量 $\vec{L}$ 是多少？请用 $m, r, R, \Omega, \hat{x}, \hat{z}$ 表示你的答案。
(c) [3] 球与地面之间的法向力 $\vec{N}$ 是多少？



Solution:



(a) The total angular velocity

$$\vec{\omega} = -\Omega\hat{z} + \omega'\hat{x}_2$$

where

$$\hat{x}_2 = \cos\theta\,\hat{x} + \sin\theta\hat{z}\,,\quad \hat{x}_3 = -\sin\theta\hat{x} + \cos\theta\hat{z}$$

The point of contact is $\vec{R} = R\hat{x}$, and the condition of pure rolling implies,

$$\vec{R}\times\vec{\omega} = 0$$
$$\Rightarrow -\Omega\hat{y} + \omega'\sin\theta\hat{y} = 0$$
$$\Rightarrow \omega' = \frac{\Omega}{\sin\theta}$$
$$\vec{\omega} = -\Omega\hat{z} + \frac{\Omega}{\sin\theta}(\cos\theta\,\hat{x} + \sin\theta\hat{z}) = \Omega\cot\theta\hat{x} = \frac{R\Omega}{r}\hat{x}$$

Alternative solution:
The CM of the sphere moves with

$$\vec{v}_{cm} = -\Omega R\hat{y}$$

If the spinning of the sphere along $\hat{x}_2$ is ω', the velocity of the contact point is

$$\vec{v}\,' = -\Omega R\hat{y} + \vec{R}\times\omega'\hat{x}_2 = (-\Omega R + R\omega'\sin\theta) = 0 \Rightarrow \omega' = \frac{\Omega}{\sin\theta}$$
$$\Rightarrow \vec{\omega} = -\Omega\hat{z} + \omega'\hat{x}_2 = \frac{R\Omega}{r}\hat{x}$$

(b)

$$I_3 = \frac{2}{5}mr^2 + m(r^2+R^2) = \frac{7}{5}mr^2 + mR^2$$

Method 1: The moment of inertia along the x-axis is $I_x = \frac{2}{5}mr^2 + mr^2 = \frac{7}{5}mr^2$

$$L_x = I_x\omega = \frac{7}{5}mrR\Omega$$

On the other hand,

$$L_z = mR^2\Omega$$

$$\vec{L} = \frac{7}{5}mrR\Omega\hat{x} - mR^2\Omega\hat{z}$$

Method 2: The moment of inertia along $\hat{x}_3$ is:

$$I_3 = \frac{2}{5}mr^2 + m(r^2+R^2) = \frac{7}{5}mr^2 + mR^2$$

$$\vec{L} = I_2\omega_2\hat{x}_2 + I_3\omega_3\hat{x}_3$$
$$\omega_2 = \vec{\omega}\cdot\hat{x}_2 = \frac{R\Omega}{r}\cos\theta\,,\qquad \omega_3 = -\frac{R\Omega}{r}\sin\theta$$
$$\Rightarrow L_x = \vec{L}\cdot\hat{x} = I_2\omega_2\cos\theta - I_3\omega_3\sin\theta = \frac{2}{5}mr^2\left(\frac{R\Omega}{r}\right)\cos^2\theta + \left(\frac{7}{5}mr^2 + mR^2\right)\frac{R\Omega}{r}\sin^2\theta = \frac{2}{5}mrR\Omega + m(r^2+R^2)\frac{R\Omega}{r}\sin^2\theta$$
$$\Rightarrow L_x = \frac{7}{5}mrR\Omega$$

$$L_z = I_2\omega_2\sin\theta + I_3\omega_3\cos\theta = \frac{2}{5}mr^2\left(\frac{R\Omega}{r}\cos\theta\sin\theta\right) + \left(\frac{7}{5}mr^2 + mR^2\right)\left(-\frac{R\Omega}{r}\sin\theta\cos\theta\right) = -m(r^2+R^2)\frac{R\Omega}{r}\sin\theta\cos\theta$$
$$= -mR^2\Omega$$

$$\Rightarrow \vec{L} = \frac{7}{5}mrR\Omega\hat{x} - mR^2\Omega\hat{z}$$

(c) The lollipop is process along the z axis with angular velocity Ω, we have

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\Omega}\times\vec{L}$$

$$\left|\tau_y\right| = R(N - mg) = \left|\frac{d\vec{L}}{dt}\right|_y = \Omega L_x = \frac{7}{5} m r R \Omega^2$$

$$\Rightarrow N = \frac{7}{5} m r \Omega^2 + mg$$

Remark: Friction will provide a torque along $\hat{z}$ direction.

It is interesting to see that the normal force is independent of R and θ.

2. In 2018, the Nobel Prize in physics was awarded to Arthur Ashkin for the creation of the "laser tweezer", a device that allows one to hold and move transparent microscopic objects with the help of light. In this device, a parallel beam of light from a laser passes through a converging lens L and hits a microparticle M, which can also be considered as a converging lens. Point F is the common focus of L and M.

The light intensity in the beam is $I = 1.00\mu\text{W/cm}^2$, the beam radius is $R = 1.00\text{cm}$, and the focal length of the lens L is $F = 10.0$ cm. Ignore the absorption and reflection of light. Speed of light in vacuum is $c = 3 \times 10^8 \text{m/s}$.

(a) [3] Find the force (magnitude and direction) due to the light acting on the lens L , in the setup shown in Fig.2a. Express the answer in terms of I, R, F and c.
(b) [2] Calculate the (numerical value) magnitude and direction of the force acting on the microparticle M, in the setup shown in Fig.2a.
Next, the left half of the lens L is covered by a diaphragm, as shown in Fig.2b.
(c) [3] Find the force (magnitude and direction) acting on the microparticle in this case. Express the answer in terms of I, R, F and c.
(d) [2] Calculate the (numerical value) magnitude and direction of the force in this case.

2018 年，诺贝尔物理学奖授予 Arthur Ashkin，以表彰其发明"激光镊子"的成就。这是一种利用光来捕获和移动透明微观物体的装置。在这装置中，激光器发出的一束平行光通过会聚透镜L，然后照射到微粒 M 上。微粒 M 也可以被假设为一个聚透镜。点 F 是 L 和 M 的共同焦点。

光束的强度为 $I = 1.00\mu\text{W/cm}^2$，光束半径为 $R = 1.00$cm，透镜 L 的焦距为 $F = 10.0$ cm。忽略光的吸收与反射。真空中的光速为 $c = 3 \times 10^8 \text{m/s}$。

(a) [3] 求透镜 L 上的光的作用力 (大小和方向) 的表达式。将答案用 I, R, F, c 表示。
(b) [2] 计算透镜 M 上的作用力的方向和数值大小。
接下来，在图 2b 所示的情况下，用光阑遮住透镜 L 的左半部分。
(c) [3] 求微粒 M 上的作用力（大小和方向）的表达式。将答案用 I, R, F, c 表示。
(d) [2] 计算此情况下作用力的方向和数值大小。

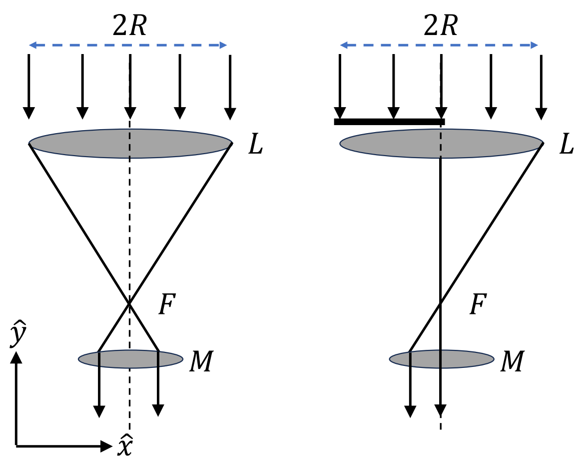


Fig. 2a Fig.2b

Solution:

(a) Consider a ring of radius r, the area is $dS = 2\pi r dr$. The change in the longitudinal momentum of photons passing through the given ring per unit time is equal to,

$$dp_\parallel = \frac{I}{c}(1-\cos\theta)dS, \qquad \text{where } \cos\theta = \frac{F}{\sqrt{F^2+r^2}}$$

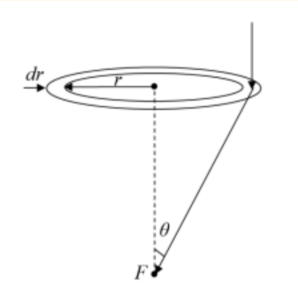
$$\Rightarrow \vec{f_\parallel} = -\hat{y}\int_0^R dp_\parallel = -\hat{y}\frac{I}{c}\int_0^R \left(1-\frac{F}{\sqrt{F^2+r^2}}\right)2\pi r dr = -\hat{y}\frac{\pi I}{c}\int_0^R \left(1-\frac{F}{\sqrt{F^2+r^2}}\right)dr^2 = -\hat{y}\frac{\pi I}{c}\left(R^2-2F\sqrt{F^2+R^2}+2F^2\right)$$

$$\approx -\hat{y}\frac{\pi I}{c}\left(2F^2+R^2-2F^2\left(1+\frac{R^2}{2F^2}-\frac{1}{8}\frac{R^4}{F^4}\right)\right) = -\hat{y}\frac{\pi I R^4}{4cF^2}$$

Alternatively, we can use the small angle approximation, $\sin\theta \approx \tan\theta = \frac{r}{F}$

$$\vec{f_\parallel} = -\hat{y}\int_0^R dp_\parallel = -\hat{y}\int_0^R \frac{I}{c}\left(1-\sqrt{1-\frac{r^2}{F^2}}\right)2\pi r dr = -\hat{y}\frac{\pi I}{c}\left(R^2-\frac{2}{3F}\left[F^3-(F^2-R^2)^{\frac{3}{2}}\right]\right) \approx -\hat{y}\frac{\pi I R^4}{4cF^2} = -2.64\times10^{-17}N\,\hat{y}$$

In this part, any method will lead to the result $\vec{f_\parallel} \approx -\hat{y}\frac{\pi I R^4}{4cF^2}$ in the limit $\frac{R}{F} \gg 1$ will get full marks.



(b) Since the foci of the lens L and the particle M coincide, when leaving the lens-particle system, the light beam propagates again parallel to the optical axis, the photon momentum is restored. As a result, the force acting on the particle M is equal in magnitude to $f_\parallel$, but is directed towards the converging lens L.

$$\vec{F} = \frac{\pi I R^4}{4cF^2}\hat{y} = 2.64\times10^{-17}N\,\hat{y}$$

If use the exact formula, we have

$$\vec{F} = \frac{\pi I}{c}\left(R^2-2F\sqrt{F^2+R^2}+2F^2\right)\hat{y} = 2.60\times10^{-17}\ N\,\hat{y}$$

(c) The transverse force is halved, $F_\parallel = \frac{\pi I R^4}{8cF^2} = 1.32\times10^{-17}$ N

There is a longitudinal component when half of the lens is covered.

$$dp_\perp = \frac{I}{c}\sin\theta\sin\beta\, r dr d\beta = \frac{I}{c\sqrt{F^2+r^2}}\sin\beta\, r^2 dr d\beta$$

Here $\sin\theta = \frac{r}{\sqrt{r^2+F^2}}$ and β is the azimuth angle in the polar coordinates.

$$p_\perp = \int dp_\perp = \frac{I}{c}\int_0^\pi d\beta\int_0^R \frac{r^2}{\sqrt{F^2+r^2}}dr\sin\beta\, d\beta = \frac{2I}{c}\int_0^R \frac{r^2}{\sqrt{F^2+r^2}}dr = \frac{I}{c}\left(R\sqrt{R^2+F^2}-F^2\,\mathrm{ArcTanh}\left(\frac{R}{\sqrt{F^2+R^2}}\right)\right)$$
$$= 2.21\times10^{-16}\mathrm{N}$$

$$p_\perp \approx \frac{I}{c}\left(R\sqrt{R^2+F^2}-\frac{F^2R}{\sqrt{R^2+F^2}}-\frac{F^2R^3}{3(R^2+F^2)^{3/2}}\right) = \frac{I}{c}\left(\frac{R^3}{\sqrt{R^2+F^2}}-\frac{R^3}{3F}\right) \approx \frac{2IR^3}{3cF}$$

Approximately, we can

$$p_\perp \approx \frac{I}{Fc}\int_0^\pi d\beta\int_0^R r^2 dr\sin\beta = \frac{2IR^3}{3cF} = 2.24\times10^{-16}N$$

(d)

$$\vec{F} = -\frac{2IR^3}{3cF}\hat{\imath} + \frac{\pi IR^4}{8cF^2}\hat{\jmath}$$

$$\vec{F} = (-2.24 \times 10^{-16}\hat{\imath} + 1.32 \times 10^{-17}\,\hat{\jmath})\ N$$

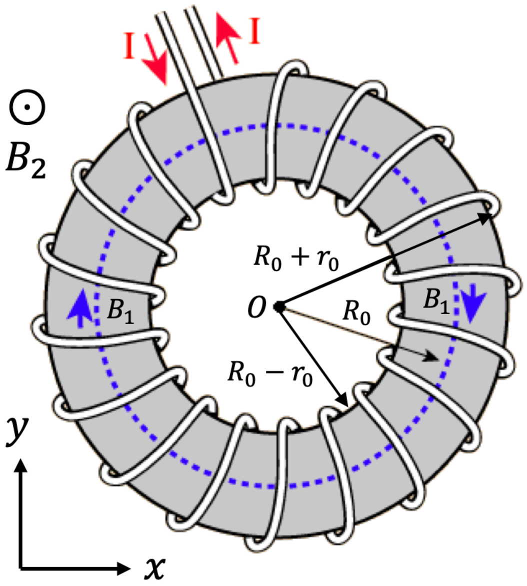
3. The toroidal cavity is designed to confine charged particles for nuclear fusion. This geometry enables charged particles to follow helical magnetic field lines, allowing them to remain suspended within the cavity without coming into contact with the walls. As depicted in the figure, consider a toroidal cavity with an outer radius R_0 and a circular cross-section of radius r_0, where $r_0 \ll R_0$. In the figure, O represents the origin.

(a) [0.5] If a wire is uniformly and tightly wound around the toroidal cavity for N turns, and the magnetic field inside the cavity is given as: $\vec{B}_1(r) = f(r)(\sin\phi\hat{x} - \cos\phi\hat{y})$, find $f(r)$ in terms of N, I, R_o, r_0 and relevant physical constants. Here r, ϕ represent the radial and angular coordinates in a polar coordinate system on the plane.

However, the diamagnetic drifty caused by the gradient of the toroidal magnetic field tends to push the particles outward, leading to a loss of confinement.
(b) [0.5] To address this, a uniform magnetic field $\vec{B}_2 = B_0\hat{z}$ is applied along the z-direction. In this uniform magnetic field $\vec{B}_2$, a particle of mass m and charge q moves in uniform circular motion with a radius R_o, find the angular frequency ω_0 of this circular motion. Express the answer in terms of q, m, B_0.
(c) [2] For a charged particle with mass m and charge q, write down the equations of motion in the r, ϕ, z directions for its motion inside the toroidal cavity with $\vec{B}_1$ and $\vec{B}_2$. Express the answer in term of the dimensionless parameter $\alpha = \frac{\mu_0 NI}{2\pi R_0 B_0}$.
(d) [2] Based on the equations of motion derived in part (C), consider the case where the charged particle is slightly perturbed from the circular motion described in part (B). Let the perturbed motion be expressed as: $r(t) = R_0 + \delta r(t), \phi(t) = -\omega_0 t + \delta\phi(t), z(t) = \delta z(t)$. Here, we assume $\delta r, \delta z \ll R_0$ and $\delta\phi \ll 1$. Derive the equations of motion that are expanded to the first order in $\delta r(t), \delta\phi(t), \delta z(t)$.
(e) [2] Given initial conditions: $\delta r(0) = 0,\ \delta\dot{r}(0) = v_0,\ \delta\phi(0) = \delta\dot{\phi}(0) = \delta z(0) = \delta\dot{z}(0) = 0$, find $\delta r(t)$ and $\delta z(t)$.
(f) [3] To prevent the charged particle from colliding with the walls of the toroidal cavity after being perturbed, what conditions must $\frac{v_0}{r_0\omega_0}$ satisfy?

3. 环形空腔的设计用于核聚变中约束带电粒子。这种几何结构使带电粒子能够沿着螺旋形磁力线运动，从而悬浮在空腔内而不接触其壁面。如图所示，考虑一个环形空腔，其外半径为 R_0，圆形横截面的半径为 r_0，其中 $r_0 \ll R_0$。在图中，O表示原点。
(a) [0.5] 如果一根电线均匀且紧密地绕环形空腔 N 匝，且空腔内的磁场表示为：$\vec{B}_1(r) = f(r)(\sin\phi\hat{x} - \cos\phi\hat{y})$。求 $f(r)$，用 N, I, R_o, r_0 和相关物理常数表示。这里 r, ϕ 是平面极坐标系中的径向和角向坐标。

然而，由于带电粒子的压力和环形磁场曲率引起的抗磁漂移，粒子倾向于向外漂移，导致约束失效。
(b) [0.5] 为了解决这个问题，沿 z-方向施加一个均匀磁场 $\vec{B}_2 = B_0\hat{z}$。在这个均匀的磁场 $\vec{B}_2$ 中，一个质量为 m、电荷为 q 的粒子以半径 R_0 进行匀速圆周运动，求其角频率 ω_0，用 q, m, B_0 表示答案。
(c) [2] 对于质量为 m、电荷为 q 的带电粒子，写出其在磁场 $\vec{B}_1, \vec{B}_2$ 作用下沿 r, ϕ, z 方向的运动方程。用无量纲参数 $\alpha = \frac{\mu_0 NI}{2\pi R_0 B_0}$ 表示答案。
(d) [2] 基于 (c) 部分中推导的运动方程，考虑带电粒子从 (b) 部分描述的圆周运动受到轻微扰动的情况。令扰动运动表示为：$r(t) = R_0 + \delta r(t), \phi(t) = -\omega_0 t + \delta\phi(t), z(t) = \delta z(t)$。 其中假设 $\delta r, \delta z \ll R_0$ 和 $\delta\phi \ll 1$。推导运动方程，并将其展开到 $\delta r(t), \delta\phi(t), \delta z(t)$ 的一阶项。
(e) [2] 给定初始条件：$\delta r(0) = 0,\ \delta\dot{r}(0) = v_0,\ \delta\phi(0) = \delta\dot{\phi}(0) = \delta z(0) = \delta\dot{z}(0) = 0$, 求 $\delta r(t)$ 和 $\delta z(t)$。
(f) [3] 为防止带电粒子在受到扰动后与环形空腔的壁面碰撞，$\frac{v_0}{r_0\omega_0}$ 必须满足什么条件？



Solution:
(a) From the Ampere's law,

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 NI \Rightarrow f(r) = \frac{\mu_0 NI}{2\pi r}$$

(b)

$$qR_0\omega_0 B_0 = m\omega_0^2 R_0$$
$$\Rightarrow \omega_0 = \frac{qB_0}{m}$$

(c) The acceleration of the particle can be written in term of cylindrical coordinate,

$$\vec{r} = r\hat{r} + z\hat{z}$$
$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$
$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$
$$\vec{B} = -\frac{\mu_0 NI}{2\pi r}\hat{\phi} + B_0\hat{z} = -\alpha B_0 \frac{R_0}{r}\hat{\phi} + B_0\hat{z}$$

The magnetic force becomes:

$$q\vec{v} \times \vec{B} = q(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}) \times \left(-\frac{\alpha B_0 R_0}{r}\hat{\phi} + B_0\hat{z}\right) = B_0 q\left(-\frac{\alpha R_0}{r}\dot{r}\hat{z} + \frac{\alpha R_0}{r}\dot{z}\hat{r} - \dot{r}\hat{\phi} + r\dot{\phi}\hat{r}\right)$$

$$m(\ddot{r} - r\dot{\phi}^2) + qB_0\left(-\frac{\alpha R_0 \dot{z}}{r} - r\dot{\phi}\right) = 0$$
$$m(2\dot{r}\dot{\phi} + r\ddot{\phi}) + qB_0\dot{r} = 0$$
$$m\ddot{z} + qB_0\left(\frac{\alpha R_0 \dot{r}}{r}\right) = 0$$

(d)

$$\delta\ddot{r} + \omega_0 R_0 \delta\dot{\phi} - \alpha\omega_0 \delta\dot{z} = 0$$
$$R_0 \delta\ddot{\phi} - \omega_0 \delta\dot{r} = 0$$
$$\delta\ddot{z} + \alpha\omega_0 \delta\dot{r} = 0$$

(e) From part d, we have

$$R_0 \delta\dot{\phi} = \omega_0 \delta r$$
$$\delta\dot{z} = -\alpha\omega_0 \delta r$$
$$\delta\ddot{r} + (1 + \alpha^2)\omega_0^2 \delta r = 0$$

$$\Rightarrow \delta r(t) = \frac{v_0 \sin(\Omega_0 t)}{\Omega_0}, \quad \Omega_0 = \sqrt{1+\alpha^2}\omega_0$$

$$\delta z(t) = -\frac{\alpha v_0 (1-\cos\Omega_0 t)}{\Omega_0\sqrt{1+\alpha^2}}$$

Also,

$$\delta\dot{\phi} = \frac{\omega_0}{R}\delta r = \frac{v_0 \sin(\Omega_0 t)}{R\sqrt{1+\alpha^2}},$$

(f)

$$\delta r^2 + \delta z^2 < r_0^2$$

$$\Rightarrow \frac{a^2}{(1+\alpha^2)}(1-\cos\Omega_0 t)^2 + \sin^2\Omega_0 t < \frac{r_0^2\omega_0^2}{v_0^2}(1+\alpha^2)$$

If $\frac{2a^2}{1+a^2} > 1$, i.e. $a > 1$, the maximum of the function is located at $\Omega_0 t = \pi$ and we have

$$\Rightarrow \frac{4a^2}{1+a^2} < \frac{r_0^2\omega_0^2}{v_0^2}(1+a^2) \Rightarrow \frac{v_0}{r_0\omega_0} < \frac{1+a^2}{2a}$$

If $\frac{2a^2}{1+a^2} < 1$, i.e. $a < 1$, the maximum is at $\cos\Omega_0 t = -a^2$ and we have

$$\Rightarrow 1 + a^2 < \frac{r_0^2\omega_0^2}{v_0^2}(1+\alpha^2) \Rightarrow \frac{v_0}{r_0\omega_0} < 1$$

4. **Harrison and the Longitude Problem**

Accurate measurement of the longitude was a long-standing problem in sea navigation. The earliest solution to this problem was to compare the local time at a location with that of a meridian. However, there was not a clock accurate enough to 'preserve' the absolute time of another location during a journey, until the inventions by the 18th century English clockmaker John Harrison.

(a) Let us start our discussion on the simplest type of clock, which is maintained by a vertically hung simple pendulum. It is made of a heavy bob of mass m, hung by a rod of length L and negligible mass, to a hinge. The timekeeping is evidently sensitive to temperature fluctuations.

(ai) [0.5] For a simple pendulum clock, would thermal expansion make it go faster or slower?

(aii) [1.5] In adjustment, Harrison proposed a modification to the rod of a pendulum, now consisting of two types of metals (of thermal expansion coefficients α_1 and α_2 respectively) by installing a central piece of metal 2 of length l'. Which of the 3 proposals in Fig. 4a can suppress the temperature fluctuations? How long should the middle piece l' of metal 2 be such that the total length of the rod L is independent of temperature fluctuations in this proposal? Express your answer in terms of L, α_1 and α_2.

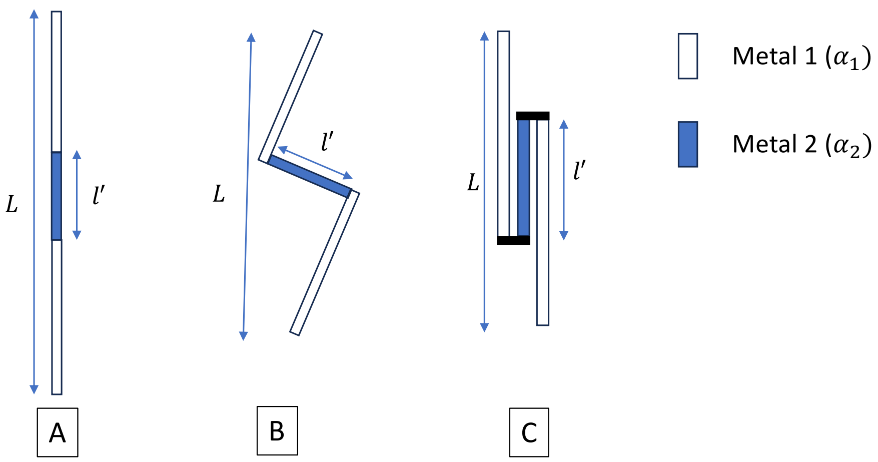
4. 哈里森与经度问题

在航海中，经度的精确测量是一个长期存在的问题。最早解决这一问题的方法是将某地的当地时间与一条子午线的时间进行比较。然而，在 18 世纪英国钟表匠约翰·哈里森的发明之前，并没有足够精确的时钟能够在航行过程中“保持”另一个地点的绝对时间。

(a) 让我们从最简单的时钟类型开始讨论，也就是由垂直悬挂的单摆维持的时钟。单摆由一个质量为 m 的重摆球组成，通过一个长度为 L 且质量可忽略的杆悬挂在一个铰链上。这种时钟的计时显然对温度变化非常敏感。

(ai) [0.5] 对于一个单摆时钟，热膨胀会使它变快还是变慢？

(aii) [1.5] 在调整中，哈里森提出对单摆的杆进行修改，将其改为由两种不同金属（热膨胀系数分别为 α_1 和 α_2）组成，并安装一段长度为 l' 的金属 2。在图 4a 的三个方案中，哪种方案可以减少温度波动？中间部分金属 2 的长度 l' 应为多少，以使杆的总长度 L 在此设计中不受温度波动的影响？请用 L, α_1, α_2 表示答案。



(b) [3] Another problem is that simple pendulums were affected by the motion of the ship it was on. To evaluate this effect, consider a ship's journey between two cities separated by $D = 96$ km shown below. Starting from rest at noon, the ship first accelerated forward at a constant rate, then maintained a constant velocity $v_0 = 27\ \text{km h}^{-1}$ until it decelerated at the same rate to arrive at the destination. The two cities are in the same time zone, and upon arrival, the local time was recorded as 16:00:00. Assuming that the pendulum clock was perfectly calibrated with real time upon departure, ~~what time did it record at the end of the journey? Please round your answer to the nearest second.~~ calculate the numerical value of the time difference between the pendulum clock and the real time at the end of the journey?
(b) [3 分] 另一个问题是单摆会受到其所在船只运动的影响。为了评估这一影响，考虑一艘船在两座城市之间的航行，这两座城市相距 $D = 96$ km，如图所示。 船从中午静止出发，首先以恒定加速度向前加速，然后以恒定速度 $v_0 = 27\ \text{km h}^{-1}$ 行驶，最后以相同的加速度减速到达目的地。这两座城市位于同一时区，船到达时记录的当地时间为 16:00:00。假设摆钟在出发时已与真实时间完美校准，~~那么摆钟在旅程结束时显示的时间是多少？请将你的答案四舍五入到最近的秒。~~计算旅程结束时钟摆钟与实际时间之间的时间差的数值。

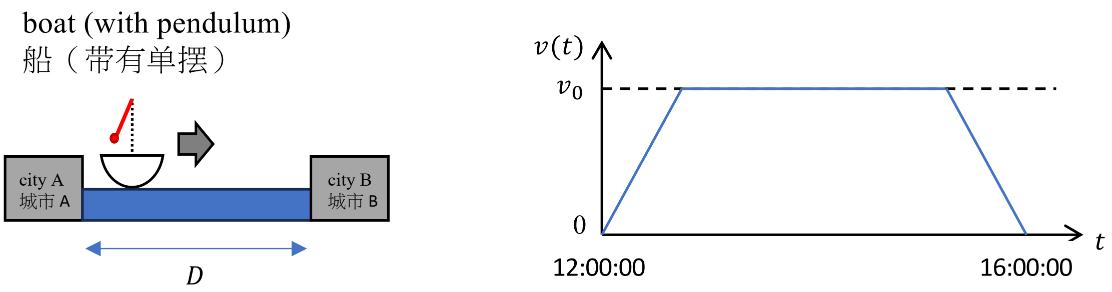


Fig. 4b

(c) To design a clock that can mitigate the effects discussed in the previous part, Harrison built the following clock, known as the 'H1'. Let us consider a simplified model of the H1 clock, shown below (on the left).

Two massive dumbbells, each of length ℓ joining two metal balls of mass M, are connected by two identical springs of stiffness k. The middle contacts ensure that the rotations of the dumbbells are always in antiphase. The oscillations of the dumbbells are used to determine time.

(ci) [0.5] Just from the simplified design, one can see that its accuracy is not affected by translational motion of the ship. Which of the following correctly summarizes the reason behind it? Select only **one** option.

A.	Spring can be made exceedingly stiff, such that elastic forces dominate by orders of magnitude over gravitational forces or any net force due to acceleration of the ship.
B.	Stabilization of the clock is actualized with the springs in the H1 clock design. Therefore, the dumbbells stay level during ship motion, and their oscillations are always perpendicular to gravity.
C.	The H1 clock is effectively equivalent to a gyroscope, which removes any bias due to translations of the boat owing to the mechanism of precession.
D.	In the non-inertial frame of the ship, the metal balls on each dumbbell experience equal and opposite torques due to translational effects, so they cancel out and do not affect oscillations.

(cii) [1.5] Determine the period T of small oscillations of the H1 clock design in terms of M, k and ℓ.
(ciii) [3] However, Harrison's H1 design still has many problems. For instance, let us consider the rocking of the ship, under the influence of waves. Let us consider a small model toy boat with the clock on board that moves on a sinusoidal landscape of the form $z = A_z \sin(2\pi x/\lambda)$ at a constant speed v_s, as shown below. What is the time-averaged new oscillation period of the clock $\langle T' \rangle$, considering that $v_s T' \ll A_z \ll \lambda$, and that the dimensions of the boat/clock are much less than A_z? Express your answer as the ratio $\frac{\langle T' \rangle - T}{T}$.

Hint: the radius of curvature R of a curve $z(x)$ is given by $\frac{1}{R} \approx \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}$.

Despite being problematic, the H1 clock proved to be fairly accurate in seafaring missions. Harrison later produced H2, H3, and H4 clocks, which were increasingly accurate, solving the longitude problem. Today, we can measure longitude to centimeter level precision owing to GPS technology.

(c) 为了设计一款能够减轻上一部分讨论中影响的时钟，哈里森制造了被称为 “H1”的时钟。我们来分析 “H1”时钟的一个简化模型，如下图所示（左侧）。

两根质量较大的哑铃（长度为 l，每根连接两颗质量为M 的金属球）通过两个刚度为 k 的相同弹簧相连。中间的接触部件确保哑铃的旋转始终处于反相状态。哑铃的振荡被用来测量时间。

(ci) [0.5] 从简化设计可以看出，其精度不会受到船只平动的影响。以下哪个选项正确总结了其背后的原因？请选择一个选项：

A. 弹簧可以设计得极其刚硬，使得弹性力比重力或因船只加速度产生的净力大出若干个数量级。
B. H1 时钟设计中的弹簧实现了时钟的稳定性。因此，在船运动期间，哑铃保持水平，其振荡始终垂直于重力方向。
C. H1 时钟实际上等效于一个陀螺仪，通过进动机制消除了因船只平动产生的任何偏差。
D. 在船只的非惯性参考系中，每根哑铃上的金属球由于平动效应受到大小相等、方向相反的力矩，因此这些力矩相互抵消，不影响振荡。

(cii) [1.5] 求出 H1 时钟设计中小振幅振荡的周期 T，用 M, k, l 表示答案。

(ciii) [3 分] 然而，哈里森的 H1 设计仍然存在许多问题。例如，我们考虑船只在波浪作用下的摇摆。设想一个装有时钟的小型玩具船，它以恒定速率 v_s 沿着形状为 $z = A_z \sin(2\pi x/\lambda)$ 的正弦波状地形移动，如下图所示。 考虑 $v_s T' \ll A_z \ll \lambda$，且船/时钟的尺寸远小于 A_z。求出时钟的新振荡周期的时间平均值 $\langle T' \rangle$，并用比值 $\frac{\langle T' \rangle - T}{T}$表示答案。 提示： 曲线 $z(x)$ 的曲率半径 R 的表达式为 $\frac{1}{R} \approx \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}$。

尽管存在问题，H1 时钟在航海任务中表现得相当准确。后来，哈里森制造了 H2、H3 和 H4 时钟，精度逐步提高，成功解决了经度问题。今天，借助 GPS 技术，我们可以将经度测量精确到厘米级别。

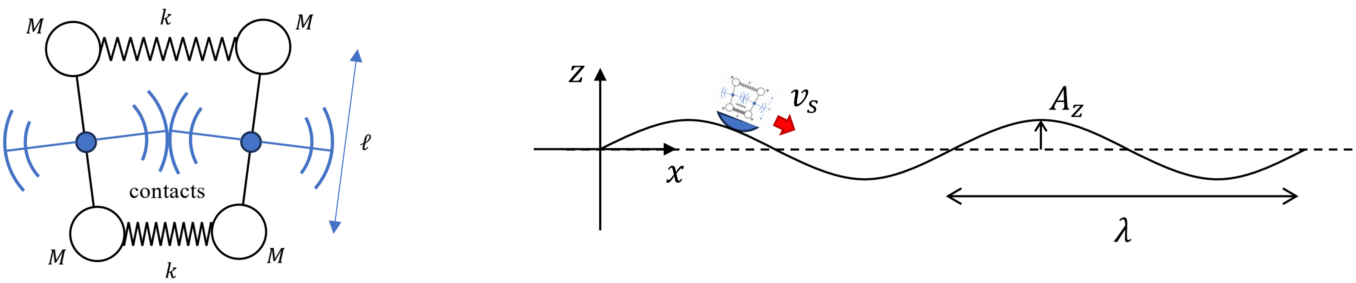


Fig. 4c

Solution:

(ai) The period of a pendulum is $T_p = 2\pi\sqrt{L/g} \propto \sqrt{L}$. Upon thermal expansion, L increases so the period increases and so the pendulum goes slower.

(aii) Let the length of the first rod (made of metal 1) be L_1. Originally, $2L_1 - l' = L$. Suppose for an increase in temperature by ΔT, the lengths of the two rods are $L_1(1+\alpha_1\Delta T)$ and $l'(1+\alpha_2\Delta T)$, we need

$$2L_1(1+\alpha_1\Delta T) - l'(1+\alpha_2\Delta T) = L \quad \Rightarrow \quad 2L_1\alpha_1\Delta T - l'\alpha_2\Delta T = 0 \quad \Rightarrow \quad l' = \frac{2\alpha_1}{\alpha_2}L_1$$

Therefore, substitute back in $L_1 = (L+l')/2$, we get

$$l' = \frac{\alpha_1}{\alpha_2 - \alpha_1}L$$

(b) Despite the clock going wrong, we are sure that the journey took $T_0 = 4$ hours. Let the common acceleration and deceleration be a. Then, by kinematic considerations,

$$t_0 = \frac{v_0}{a}$$

$$\frac{1}{2}a\left(\frac{v_0}{a}\right)^2 \times 2 + \left(T_0 - \frac{2v_0}{a}\right)v_0 = D$$

$$\Rightarrow v_0\left(\frac{v_0}{a} + T_0 - \frac{2v_0}{a}\right) = D$$

$$\Rightarrow \frac{v_0}{a} = T_0 - \frac{D}{v_0}$$

$$\Rightarrow a = \frac{v_0}{T_0 - D/v_0}$$

When the pendulum is oscillating in an accelerated frame, the bob experiences a fictitious force of ma, so the net effective gravity it experiences is $g_{\text{eff}} = \sqrt{g^2 + a^2}$. The effective period T_{eff} is:

$$T_{\text{eff}} = \frac{2\pi\sqrt{L}}{(g^2 + a^2)^{1/4}} \approx 2\pi\sqrt{\frac{L}{g}}\left(1 - \frac{1}{4}\frac{a^2}{g^2}\right)$$

Hence, the ship's clock is faster than expected, and since it is a second-order effect, the same change in period occurs during acceleration and deceleration.

Suppose the pendulum oscillates N cycles during the acceleration, we have

$$N \times T_{\text{eff}} = t_0 \Rightarrow N = \frac{t_0}{T_{\text{eff}}}$$

The total difference in time is:

$$\Delta t = N(T_0 - T_{\text{eff}}) \times 2 = \left(\frac{T_0}{T_{\text{eff}}} - 1\right) \times \left(\frac{v_0}{a}\right) \times 2 \approx \frac{1}{4}\frac{a^2}{g^2}\left(\frac{v_0}{a}\right) \times 2 = \frac{v_0^2}{2g^2\left(T_0 - \frac{D}{v_0}\right)} \approx 1.8 \times 10^{-4}\ \text{s}$$

(ci) The answer is D.
(cii) Let the rotation angle of the dumbbell with respect to the 'vertical' be θ. In the state as shown in the figure, the net torque on the dumbbell is:

$$\tau = \left(\frac{\ell}{2}\cos\theta \times 2\right)\left(k\frac{\ell}{2}\sin\theta\right) - \left(-\frac{\ell}{2}\cos\theta \times 2\right)\left(k\frac{\ell}{2}\sin\theta\right) = \frac{\sin 2\theta}{2}k\ell^2$$

The moment of inertia of the dumbbell with respect to its center of mass is $I = M(\ell/2)^2 \times 2 = M\ell^2/2$. Hence, the equation of motion of the dumbbell is:

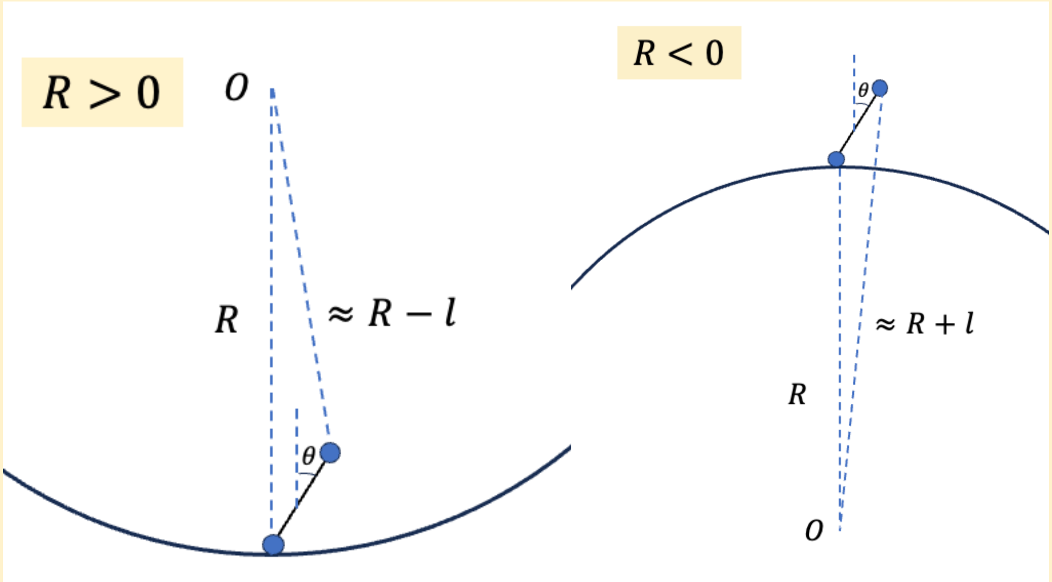
$$\tau = I\ddot{\theta} \quad \Rightarrow \quad \frac{M\ell^2}{2}\ddot{\theta} = -\frac{\sin 2\theta}{2}k\ell^2 \quad \Rightarrow \quad \ddot{\theta} = -\frac{2k}{M}\theta$$

Hence, the period of oscillations is:

$$T = 2\pi\sqrt{\frac{M}{2k}}$$

(ciii) In the non-inertial reference frame of the boat, the effect leading to a change in oscillation period is a torque-inducing unevenness of centrifugal force acting on the dumbbells. First, the radius of curvature of the sinusoidal track can be found as follows:

$$\frac{1}{R} \approx \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = A_z\left(\frac{2\pi}{\lambda}\right)^2 \sin\left(\frac{2\pi}{\lambda}x\right)$$



For $R > 0$, the centrifugal force induces an extra torque τ_f which tends to align the dumbbell with its 'vertical', given by

$$\tau_f = \Delta F\left(\frac{\ell}{2}\sin\theta\right) = M(\omega^2(R-l) - \omega^2 R)\left(\frac{l}{2}\sin\theta\right) = -\frac{M\omega^2 l^2}{2}\theta = -\frac{Mv_s^2 l^2}{2R^2}\theta$$

For $R < 0$,

$$\tau_f = \Delta F\left(\frac{\ell}{2}\sin\theta\right) = M\big(\omega^2|R| - \omega^2(R+l)\big)\left(\frac{\ell}{2}\sin\theta\right) \approx -\frac{Mv_s^2\ell^2}{2R^2}\theta$$

Hence, we can rewrite the equation of motion as

$$\ddot{\theta} = -\frac{1}{M}\left(2k + \frac{Mv_s^2}{R^2}\right)\theta$$

And so, the fractional change in period is:

$$\frac{\langle T'\rangle - T}{T} = \left\langle \sqrt{\frac{2k}{2k + Mv_s^2/R^2}} - 1 \right\rangle \approx \left\langle -\frac{Mv_s^2}{4kR^2}\right\rangle \approx -\frac{Mv_s^2}{4k}A_z^2\left(\frac{2\pi}{\lambda}\right)^4\left\langle \sin^2\left(\frac{2\pi}{\lambda}x\right)\right\rangle = -\frac{2\pi^4 A_z^2 M v_s^2}{k\lambda^4}$$

Thanks Terry Lam for providing this question.

~ End of Part 1 卷-1 完 ~