Pan Pearl River Delta Physics Olympiad 2025

2025 年泛珠三角及中华名校物理奥林匹克邀请赛

Sponsored by Institute for Advanced Study, HKUST

香港科技大学高等研究院赞助

Simplified Chinese Part-2 (Total 2 Problems, 60 Points)

简体版卷-2（共2题，60分）

(1:30 pm – 5:00 pm, 2 February 2025)

All final answers should be written in the **answer sheet**.
所有最后答案要写在**答题纸**上。

All detailed answers should be written in the **answer book**.
所有详细答案要写在**答题簿**上。

There are 2 problems. Please answer each problem starting on a **new page**.
共有 2 题，每答 1 题，须采用**新一页纸**。

Please answer on each page using a **single column**. Do not use two columns on a single page.
每页纸请用**单一直列**的方式答题。不可以在一页纸上以双直列方式答题。

Please answer on **only one page** of each sheet. Do not use both pages of the same sheet.
每张纸**单页**作答。不可以双页作答。

Rough work can be written in the answer book. Please cross out the rough work after answering the questions. No working sheets for rough work will be distributed.
草稿可以写在答题簿上，答题后要在草稿上划上交叉，不会另发草稿纸。

If the answer book is not enough for your work, you can raise your hand. Extra answer books will be provided. Your name and examination number should be written on all answer books.
考试中答题簿不够可以举手要，所有答题簿都要写下姓名和考号 。

At the end of the competition, please put the **question paper and answer sheet** inside the answer book. If you have extra answer books, they should also be put inside the first answer book.
比赛结束时，请把考卷和答题纸夹在答题簿里面，如有额外的答题簿也要夹在第一本答题簿里面。

Problem 1: Golfer's Nightmare [30 pt] 问题 1: 高尔夫球手的噩梦 [30 分]

What happens when a golf ball rolls along the inner vertical wall of the cylindrical hole under gravity? Normally, one thinks the ball will go in and never come back up. However, it is often observed that the ball first rolls down along the wall and but then it rolls back up without touching the bottom of the hole.
在重力作用下，高尔夫球沿圆柱形洞的内垂直壁滚动时会发生什么？通常，人们认为球会滚进去而不会再滚上来。然而，常常观察到球先沿着墙壁滚下，然后又滚上来而不接触洞底。

Some mathematics Identities may be useful in this problem:
在这个问题中，这些数学恒等式可能有用:

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$
$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\,\vec{B}-(\vec{A}\cdot\vec{B})\,\vec{C}$$

To understand this phenomenon, we consider a ball rolls without slipping along the vertical wall of a cylindrical hole at all time. The hole has a depth H and radius R. The ball is spherically symmetrical, has a mass m, a radius r and a moment of inertia $I = Kmr^2$, where K is a numerical constant that depends on the mass distribution inside the ball, such as $K = 2/5$ for a uniform sphere and $K = 2/3$ for a spherical thin shell, etc. Let

$\vec{r}$ be the position vector of the center of the ball;

$\vec{\omega}$ be the angular velocity vector of the rotation of the ball;

$\vec{g}$ be the constant acceleration due to gravity pointing towards $-z$ direction, i.e. $\vec{g} = -g\,\hat{e}_z$

We ignore air friction in this problem.
为了理解这种现象，我们考虑一个球在圆柱形洞的垂直壁上始终无滑动地滚动。洞的深度为 H，半径为 R。球是球对称的，具有质量 m，半径 r及惯性矩为$I = Kmr^2$，其中 K 是一个数值常数，取决于球内部的质量分布，例如，对于均匀球体$K = 2/5$，对于球形薄壳$K = 2/3$等。设

$\vec{r}$ 为球心的位置矢量；

$\vec{\omega}$ 为球的旋转角速度矢量；

$\vec{g}$ 为重力引起的恒定加速度，指向 $-z$方向， 即 $\vec{g} = -g\,\hat{e}_z$。

在这个问题中，我们忽略空气摩擦。

The situation is shown in Figure 1 below in cylindrical coordinates. Point C is the center of the ball. Point P is the contact point between the ball and the wall. The basis vectors $\hat{e}_\rho$ and $\hat{e}_\phi$ are the basis vectors in cylindrical coordinates (ρ, ϕ, z), where ρ is the perpendicular distance from C to the z-axis, ϕ is the azimuthal angle measured from x-axis and z is the vertical distance of C from the xy-plane containing the origin O. Given the basis vectors in cylindrical coordinates as

$$\hat{e}_\rho = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}, \qquad \hat{e}_\phi = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}, \qquad \hat{e}_z = \hat{k}$$

where $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are the unit vectors of the Cartesian coordinates along x, y and z axis, respectively.
情况如图 1 所示，使用圆柱坐标系表示。点C 是球的中心。点 P 是球与墙的接触点。基矢量 $\hat{e}_\rho$ 和 $\hat{e}_\phi$是圆柱坐标系 (ρ, ϕ, z) 中的基矢量，其中 ρ 是从 C 到 z 轴的垂直距离， ϕ 是从x轴测量的方位角，z 是 C 到包含原点 O 的 xy 平面的垂直距离。给定圆柱坐标系中的基矢量为：

$$\hat{e}_\rho = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}, \qquad \hat{e}_\phi = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}, \qquad \hat{e}_z = \hat{k}$$

其中 $\hat{\imath}, \hat{\jmath}$ 和 $\hat{k}$ 分别是笛卡尔坐标系中沿x, y和z轴的单位矢量。

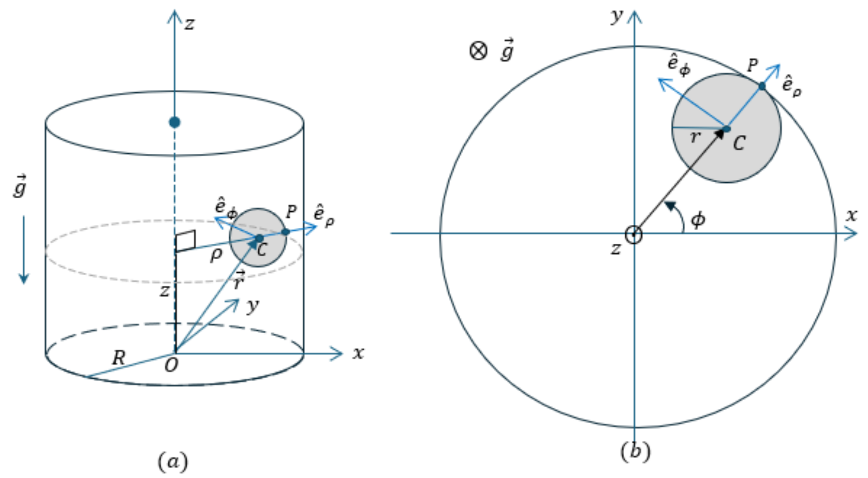


Figure 1. (a) The ball and the cylinder. (b) Top view of the situation.
图1. (a) 球和圆柱体。
(b) 顶视图

PART I: No friction between the ball and the wall. 第一部分：球和墙之间没有摩擦。

Firstly, we begin with a case where the wall and the ball are frictionless.

首先，我们从墙和球没有摩擦的情况开始。

Suppose the force exerting on the ball by the wall at the contact point P is $\vec{F} = -N\hat{e}_\rho$ which is only the normal force.

假设在接触点 P 处，墙对球施加的力为 $\vec{F} = -N\hat{e}_\rho$，这只是法向力。

(a)	Write down the position vector of point C in terms of m, g, R, r, ϕ, z, their derivatives and basis vectors in cylindrical coordinates. 用 m, g, R, r, ϕ, z 及其导数和圆柱坐标系中的基矢量表示点 C 的位置矢量。	1 Points 1 分
(b)	Given the initial condition: at time $t = 0$, $z(0) = H$, $\phi(0) = 0$, $\dot{z}(0) = 0$ and $\dot{\phi}(0) = \Omega > 0$. Find ϕ, z and N as a function of t. 给定初始条件：在时间 $t = 0$ 时，$z(0) = H$， $\phi(0) = 0$， $\dot{z}(0) = 0$ 和 $\dot{\phi}(0) = \Omega > 0$。 求 ϕ, z 和 N 作为 t 的函数。	2 Points 2 分

PART II: Rolling without slipping on the wall 第 II 部分：沿墙无滑动滚动

Setting up the problem 问题设置

As we see in PART I, without friction, obviously, the ball will only accelerate downward. Now, we consider the ball is rolling without slipping on the wall surface. Suppose the forces exerting on the ball by the wall at the contact point P include the normal force and the static friction as

正如我们在第一部分中看到的那样，没有摩擦时，球显然只会向下加速。现在，我们考虑球在墙面上无滑动地滚动。假设墙在接触点 P 处对球施加的力包括法向力和静摩擦力，如下所示：

$$\vec{F} = -N\hat{e}_\rho + F_\phi\hat{e}_\phi + F_z\hat{e}_z$$

Using cylindrical coordinates, let the position of the center of mass (CM) of the ball as

使用圆柱坐标系，设球的质心位置为

$$\vec{r} = \rho\hat{e}_\rho + z\hat{e}_z$$

and the angular velocity vector of the ball with respect to the CM as

及相对于质心的球角速度矢量为

$$\vec{\omega} = \omega_\rho\hat{e}_\rho + \omega_\phi\hat{e}_\phi + \omega_z\hat{e}_z$$

Answer the following questions in terms of $m, g, R, r, \phi, z, N, I, F_\phi, F_z, \omega_\rho, \omega_\phi, \omega_z$ and their derivatives.

在必要时，用 $m, g, R, r, \phi, z, I, N, F_\phi, F_z, \omega_\rho, \omega_\phi, \omega_z$及其导数回答以下问题。

(c)	Find the equations of motion describing the CM of the ball. 找出描述球质心运动的方程。	3 Points 3 分
(d)	Write down a set of differential equations for ω_ρ, ω_ϕ and ω_z. 写出 ω_ρ， ω_ϕ 和 ω_z的一组微分方程。	3 Points 3 分
(e)	Write down equations of the rolling without slipping condition. 写出无滑动滚动条件的方程。	2 Points 2 分

Solving the problem 解题

(f)	Find, from the equations of motion, that the rate of change of the z-component (vertical) of the total angular momentum of the ball with respective to the origin O. 从运动方程中找出球相对于原点 O 的总角动量在 z 分量（垂直）的变化率。	3 Points 3 分

Consider the initial condition that at the top of the hole as at time $t = 0$, $z(0) = H$, $\phi(0) = 0$, $\dot{z}(0) = 0$ and $\dot{\phi}(0) = \Omega > 0$.
考虑初始条件：在洞的顶部，即时间 $t = 0$时 ， $z(0) = H$，$\phi(0) = 0$，$\rho = R - r$，$\dot{z}(0) = 0$ 且 $\dot{\phi}(0) = \Omega > 0$。

(g)	Show that the vertical motion of the ball is a simple harmonic motion. 证明球的垂直运动是简谐运动。	3 Points 3 分

Since the vertical motion is a SHM, if the golf ball has a tiny non-zero initial downward motion at the top of the hole, the ball will first roll down and, at the time that the golfer is happy about finishing the hole, the ball will come back up and out again. That is the golfer's nightmare!
由于垂直运动是简谐运动，如果高尔夫球在洞的顶部有一个微小的(非零)初始向下运动，球将首先沿着洞壁向下滚动，当高尔夫球手因为球进了洞而高兴时，球会再次向上并走出洞来。这就是高尔夫球手的噩梦！

Consider the ball is a uniform solid sphere, i.e. $K = \frac{2}{5}$.

考虑球是一个均匀实心球，即 $K = \frac{2}{5}$。

(h)	Find the angular frequency of the vertical SHM, Ω_z, in terms of the initial angular velocity of the CM of the ball around the hole, Ω. 找出垂直简谐运动的角频率 Ω_z，用球质心绕洞的初始角速度 Ω 来表示。	1 Points 1 分
(i)	Find minimum depth of the hole H_m such that the ball will not touch the bottom of the hole in terms of g and Ω. 找出洞的最小深度H_m，使得球不会触碰洞底，用 g 和 Ω 来表示。	1 Points 1 分

Interpretation of the phenomenon 现象解释

Where does the energy go? 能量去哪了？

(j)	Using energy conservation, find the magnitude of angular velocity ω_ρ, ω_ϕ and ω_z of the ball at the lowest point in its vertical motion. 使用能量守恒，找出球在其垂直运动最低点时的角速度分量ω_ρ、ω_ϕ 和 ω_z 的大小。	2 Points 2 分

What makes the ball go upward after getting into the hole? 是什么让球在进入洞后向上运动？

To help explaining this phenomenon, one can investigate the motion of the ball in a reference frame S' (x', y', z') that rotates about the z-axis with the same angular velocity $\Omega\hat{e}_z$ as the motion of the ball around the hole. In this S' frame, the ϕ' value of the position of the ball is fixed at $\phi' = 0$.
为了帮助解释这一现象，可以研究在一个参考系 S' (x', y', z') 中球的运动，该参考系绕 z 轴以与球绕洞运动相同的角速度 $\Omega\hat{e}_z$ 旋转。在这个 S' 系中，球的位置的 ϕ' 值固定为
$\phi' = 0$。

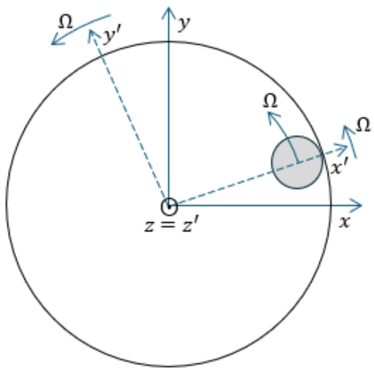


Figure 2: Top view of the axes of the rotating frame S'. The fixed and rotating frames share the same origin and vertical axis: z and z'.
图 2：旋转参考系 S' 的轴的俯视图。固定和旋转参考系共享相同的原点和垂直轴： z 和 z' 轴。

In this rotating frame, there are three fictitious forces including the centrifugal force, the Euler force $\vec{F}_E = -m\dot{\vec{\Omega}} \times \vec{r}'$ and the Coriolis force $\vec{F}_C = -2m\vec{\Omega} \times \frac{d}{dt}\vec{r}'$, where $\vec{r}'$ is the position vector in S' frame.

在这个旋转参考系中，有三个虚拟力，包括离心力、欧拉力 $\vec{F}_E = -m\dot{\vec{\Omega}} \times \vec{r}'$ 和科里奥利力 $\vec{F}_C = -2m\vec{\Omega} \times \frac{d}{dt}\vec{r}'$ ，其中 $\vec{r}'$ 是 S' 系中的位置矢量。

(k1)	Find the z-component of the centrifugal force of the entire ball. 找出整个球的离心力的 z 分量。	3 Points 3 分
(k2)	Find the z-component of the Euler force of the entire ball. 找出整个球的欧拉力的 z 分量。	
(k3)	Find the z-component of the Coriolis force of the entire ball. 找出整个球的科里奥利力的 z 分量。	

The rotation of the ball about the y'-axis (or the ϕ-componet in cylindrical coordinates) is coupled with the vertical motion of the ball when it is rolling without slipping. Therefore, knowing the torque of this rotation will give us insight into why the ball can move up and down under gravity. Let the angular velocity of the ball in this rotating frame as $\vec{\omega}'$.
当球无滑动地滚动时，球绕 y' 轴的旋转（或圆柱坐标中的 ϕ 分量）与球的垂直运动耦合。因此，了解这种旋转的力矩将帮助我们理解为什么球在重力作用下可以上下运动。设球在这个旋转参考系中的角速度为 $\vec{\omega}'$。

(l1)	Find the torque with respect to the CM of the ball due to centrifugal force. 找出离心力相对于球质心的力矩。	6 Points 6 分
(l2)	Find the torque with respect to the CM of the ball due to Euler force. 找出欧拉力相对于球质心的力矩。	
(l3)	Find the torque with respect to the CM of the ball due to Coriolis force. 找出科里奥利力相对于球质心的力矩。	
(l4)	Identify the fictitious force and condition that yields a torque which corresponds to the ball rolling up the wall. 找出产生力矩的虚拟力和条件，该力矩对应于球沿墙向上滚动的情形。	

Solution:

(a)

$\vec{r} = (R - r)\hat{e}_\rho + z\,\hat{e}_z$

(b)

In general, in cylindrical coordinates $\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z$

$\rho = R - r$ and $\dot{r} = \dot{R} = 0$

$$\vec{F}_{net} = (-N, 0, -mg)$$

According to Newton's 2 nd Law:

$$\begin{cases} m(R-r)\dot{\phi}^2 = N \\ m(R-r)\ddot{\phi} = 0 \\ \ddot{z} = -g \end{cases}$$

(c)

$$m(R-r)\dot{\phi}^2 = N \quad (1a)$$
$$m(R-r)\ddot{\phi} = F_\phi \quad (1b)$$
$$m\ddot{z} = F_z - mg \quad (1c)$$

(d)

Take moment about the CM of the ball.

$$\vec{L} = I\begin{pmatrix}\omega_\rho \\ \omega_\phi \\ \omega_z\end{pmatrix}$$

$$\dot{L} = I\begin{pmatrix}\dot{\omega}_\rho \\ \dot{\omega}_\phi \\ \dot{\omega}_z\end{pmatrix} + \begin{pmatrix}0 \\ 0 \\ \dot{\phi}\end{pmatrix} \times\ I\begin{pmatrix}\omega_\rho \\ \omega_\phi \\ \omega_z\end{pmatrix} = I\begin{pmatrix}\dot{\omega}_\rho \\ \dot{\omega}_\phi \\ \dot{\omega}_z\end{pmatrix} + \begin{pmatrix}-\dot{\phi}\, I\omega_\phi \\ \dot{\phi}\, I\omega_\rho \\ 0\end{pmatrix}$$

Torque

$$\vec{\tau} = r\hat{e}_\rho \times \vec{F} = \begin{pmatrix}\omega_\rho \\ \omega_\phi \\ \omega_z\end{pmatrix}$$

We have

$$\dot{\vec{L}} = \vec{\tau}$$

$$I\dot{\omega}_\rho - I\dot{\phi}\omega_\phi = 0 \tag{2a}$$

$$I\dot{\omega}_\phi + I\dot{\phi}\omega_\rho = -rF_z \tag{2b}$$

$$I\dot{\omega}_z = rF_\phi \tag{2c}$$

(e)

$$(R-r)\dot{\phi} + r\omega_z = 0 \tag{3a}$$

$$\dot{z} - r\omega_\phi = 0 \tag{3b}$$

(f)

Combining (1b) and (2c), we have

$$I\dot{\omega}_z - mr(R-r)\ddot{\phi} = 0 \tag{4}$$

Differentiating (3a) we have

$$(R-r)\ddot{\phi} = -r\dot{\omega}_z \tag{5}$$

Using (4) and (5), we find that

$$I\dot{\omega}_z + mr^2\dot{\omega}_z = 0$$

$$\Rightarrow \dot{\omega}_z = 0$$

$$\Rightarrow \omega_z = \text{constant}$$

Together with (3a), we have

$$\dot{\phi} = \text{constant}$$

Therefore, the z-component of the total angular momentum of the ball w.r.t. O

$$L_{z/O} = m(R-r)^2\dot{\phi} + I\omega_z = \text{constant}$$

(g)

From (2a) and (3b),

$$\dot{\omega}_\rho = \frac{\dot{\phi}\dot{z}}{r} = \frac{\Omega}{r}\dot{z}$$

With the initial condition, $\omega_\rho = 0 \; when \; z = H$, we have

$$\omega_\rho = \frac{\Omega}{r}(z - H) \tag{6}$$

Combining (1c), (2b), (3b) and (6) together with $I = Kmr^2$, we have

$$\dot{\omega}_\phi = -\frac{K}{K+1}\Omega\omega_\rho - \frac{g}{r}\frac{1}{K+1}$$

$$\frac{\ddot{z}}{r} = -\frac{K}{K+1}\frac{\Omega^2}{r}(z - H) - \frac{g}{r}\frac{1}{K+1}$$

$$\ddot{z} = -\frac{K}{K+1}\Omega^2\left(z - H + \frac{g}{\Omega^2 K}\right)$$

The vertical motion is a SHM.

(h) Angular frequency = $\Omega_v = \sqrt{\frac{K}{K+1}}\,\Omega$.

(i) the equilibrium is at $z_0 = H - \frac{g}{\Omega^2 K}$. The amplitude of the vertical oscillation is $A = \frac{g}{\Omega^2 K}$.

The minimum H is $H_{\min} = 2A = \frac{2g}{\Omega^2 K}$.

(j) The gravitational potential energy is converted into the rotational energy along $\hat{e}_\rho$. Since ω_z is constant: $\omega_z = -\frac{R-r}{r}\Omega$

and $\omega_\phi = \frac{\dot{z}}{r} = 0$ at the lowest point, the gravitational potential energy can only be converted into $\frac{1}{2}I\omega_\rho^2$ as $\omega_\rho^2 = \frac{\Omega^2}{r^2}(z - H)^2$ increases as z decreases from H. At the lowest point $z = H - 2A$,

$$\omega_\rho = \frac{5g}{\Omega r} \quad \text{or} \quad \frac{2g}{K\Omega r}$$

(k1) Centrifugal force = $-m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}')$. Since $\vec{\Omega} = \Omega\hat{e}_{z'}$, the z-component is zero.

(k2) Euler force is zero as $\dot{\Omega} = 0$

(k3) Coriolis force is zero along z'-axis since $(\vec{\Omega} \times \vec{r}') \perp \hat{e}_{z'}$

(l1) Centrifugal force is radially outward; thus, its torque is zero.

(l2) It is zero as Euler force is zero as a whole.

(l3) The velocity of a point in the ball in the rotating frame is due to the spinning $\vec{\omega}'$ of the ball only because the CM does not move w.r.t. the rotating frame. We have

$$\vec{v}' = \frac{d}{dt}\vec{r}' = \vec{\omega}' \times \vec{r}_c{}'$$

where $\vec{r}_c{}'$ is the position vector measured from the CM of the ball.

$$\vec{\tau} = -2\int dm\, \vec{r}_c{}' \times \left(\vec{\Omega} \times (\vec{\omega}' \times \vec{r}_c{}')\right)$$

$$= -2\int dm\left((\vec{\Omega}\cdot\vec{r}_c{}')(\vec{r}_c{}' \times \vec{\omega}') - (\vec{\Omega}\cdot\vec{\omega}')(\vec{r}_c{}' \times \vec{r}_c{}')\right)$$

$$= -2\int dm\,(\vec{\Omega}\cdot\vec{r}_c{}')(\vec{r}_c{}' \times \vec{\omega}')$$

$$= -2\int \rho dx_c' dy_c' dz_c' \ \Omega\, z_c' \left(\vec{r}_c{}' \times \vec{\omega}'\right)$$

The integration on x_c', y_c' and $z_c{}'$ is over the volume of the ball from its center.

Since the integration limit of $z_c{}'$ is symmetrical about $z_c{}' = 0$, only even function of $z_c{}'$ in the integrant will be non-zero. Therefore, only the terms linear to z' in $(\vec{r}' \times \vec{\omega}')$ will survive. That is

$$z_c' \omega_{x'} \hat{e}_{y'} - z_c' \omega_{y'} \hat{e}_{x'}$$

The torque becomes

$$\vec{\tau} = -2\int \rho dx_c' dy_c' dz_c' \,\Omega\, z_c' \left(z_c' \omega_{x'} \hat{e}_{y'} - z_c' \omega_{y'} \hat{e}_{x'}\right)$$

$$= -2\int \rho dx_c' dy_c' dz_c' \, z'^2 \left(\Omega\, \omega_{x'} \hat{e}_{y'} - \Omega\, \omega_{y'} \hat{e}_{x'}\right)$$

Using $\vec{\Omega} = \Omega\, \hat{e}_{z'}$, we have

$$\vec{\tau} = -2\int \rho dx_c' dy_c' dz_c' \, z_c'^2 \left(\vec{\Omega} \times \vec{\omega}'\right)$$

The integral over the volume of the ball with respect to the center of mass can be evaluated by cutting the ball into layers of circular disk with radius $(r^2 - z_c'^2)$ and thickness $dz_c{}'$

$$\int \rho dx_c' dy_c' dz_c' \, z_c'^2$$

$$= \rho \int \pi\left(r^2 - z_c'^2\right) dz_c' \, z_c'^2$$

$$= \rho \int_{-r}^{r} \pi\left(r^2 z_c'^2 - z_c'^4\right) dz_c'$$

$$= \frac{m}{\frac{4}{3}\pi r^3}\, \pi\, 2\left(\frac{r^5}{3} - \frac{r^5}{5}\right)$$

$$= \frac{1}{5} m r^2$$

So the torque is

$$\vec{\tau} = \frac{2}{5} m r^2 \ \left(\vec{\omega}' \times \vec{\Omega}\right) = I\left(\vec{\omega}' \times \vec{\Omega}\right)$$

(I4)

For non-zero ω_x', the Coriolis force yields a torque along y' direction which gives the ball a rolling motion up and down along the wall.

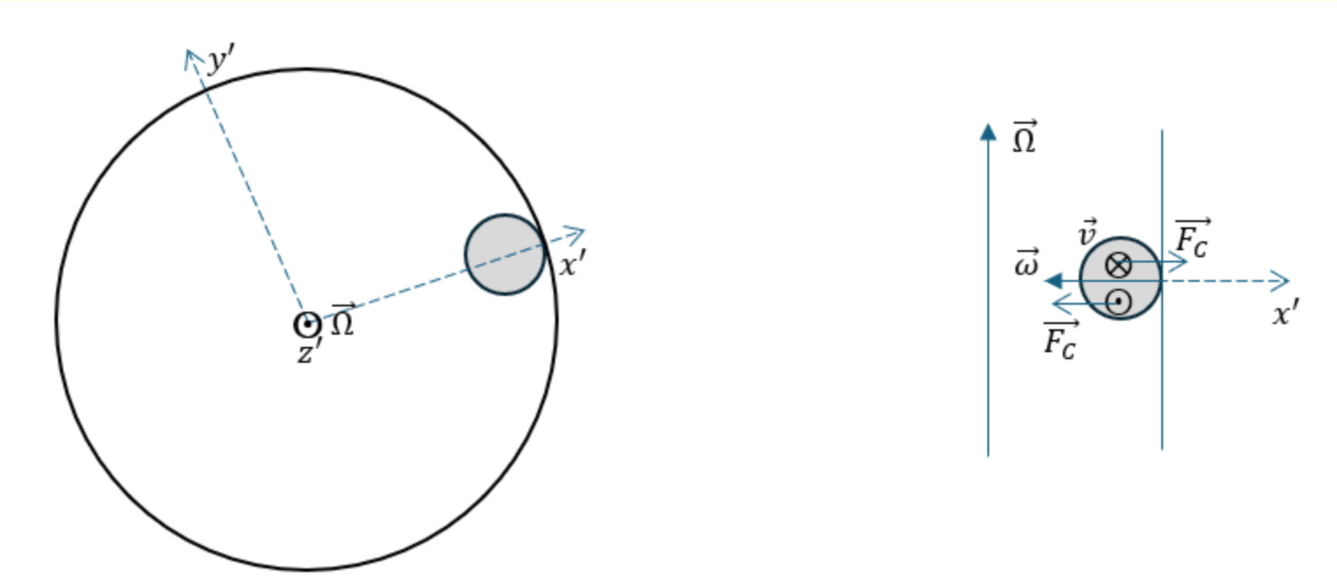
Qualitatively:

Coriolis force yields a torque along y' direction which results in a vertical upward motion of the ball.

All fictitious forces are zero to the CM. Centrifugal and Euler force yield zero torque to the ball. However, the Coriolis force provides non-zero torque to the ball and the magnitude of the torque is proportional to ω_ρ.

The Coriolis force is $\vec{v} \times \vec{\Omega}$. At different points on the ball, the velocities are different. Since $\vec{\Omega} = \Omega \hat{e}_z$, the velocity along z-axis can be ignored.

Consider the rotation along $-\hat{e}_\rho$.



The Coriolis forces on the top and bottom part of the ball are in opposite direction. These forces give the ball a net torque along $\hat{e}_\phi$. Rotation along $\hat{e}_\phi$ is coupled with the vertical motion of the ball under the rolling without slipping condition. Therefore, the Coriolis force imposed on ω_ρ gives rise to the vertical motion of the ball.

Problem 2: Generation of ultrashort electromagnetic pulse[30 pts]
问题 2: 超短电磁脉冲[30 分]

The Nobel Prize in Physics 2018 & 2023 were awarded to pioneers who contributed to "Method of generating high-intensity, ultra-short optical pulses" and "Generation of attosecond pulses of light for the study of electron dynamics in matter". Attosecond pulse refers to electromagnetic field with a duration on the order of 10^{-18} second. The advent of attosecond technique has made possible the study of ultrafast dynamics in physical, chemical and biological systems at a record high temporal resolution. Thus far, the most widely used method to generate attosecond pulse (Nobel Prize in Physics 2023) is to rely on the interaction of gas molecules and intensive femtosecond laser pulse (Nobel Prize in Physics 2018). In this question, we will explore some important aspects of the short pulse generation.

2018 年和 2023 年诺贝尔物理学奖授予了在"产生高强度、超短光脉冲的方法"和"产生用于研究物质电子动力学的阿秒光脉冲"方面做出贡献的物理学家。阿秒脉冲是指持续时间约为 10^{-18} 秒的电磁场。阿秒技术的出现使得以高时间分辨率研究物理、化学和生物系统中的超快动力学成为可能。迄今为止，最广泛使用的产生阿秒脉冲（2023 年诺贝尔物理学奖）的方法是依靠气体分子与强飞秒激光脉冲的相互作用（2018 年诺贝尔物理学奖）。在这个问题中，我们将探讨产生超短电磁脉冲的基本物理过程。

The following identity may be useful:
在这个问题中，这个数学恒等式可能有用:

$$\int_{-\infty}^{\infty} e^{-a\omega^2} e^{-i\omega t} d\omega = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{t^2}{4a}\right)$$

Physical constants 物理常数:

Electric charge 元电荷: $e = 1.60 \times 10^{-19}$ C

Electron mass 电子质量 $m_e = 9.11 \times 10^{-31}$ kg

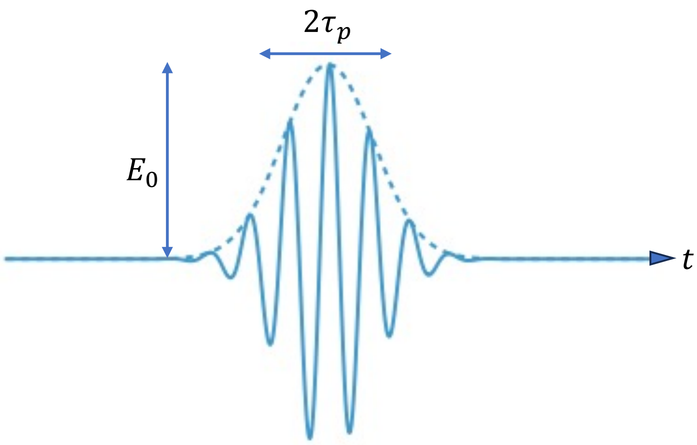
Speed of light in vacuum 真空中的光速 $c = 3.00 \times 10^8$ m/s

Planck constant 普朗克常数 $h = 6.63 \times 10^{-34}$ Js

Part A: Ultrashort laser pulse 超短激光脉冲

The simplest form of pulsed electromagnetic field is a sinusoidal wave (center frequency ω_0) dressed in a Gaussian profile with peak E_0 and a standard deviation of $\tau_p/\sqrt{2}$.

脉冲电磁场的最简单形式是具有高斯包络的中心频率为ω_0的波动振荡，峰值为 E_0，标准差为 $\tau_p/\sqrt{2}$ 。



A1	Please find the complex expression of a Gaussian signal in the time domain. Note that a Gaussian laser pulse must have its electric field averages to zero over time. Show your answer satisfy the constraint above. 请找出高斯信号在时域中的复数表达式。请注意，随着时间的推移，高斯激光脉冲的电场平均值	3 Points 3 分

	必须为零。请证明你的答案满足上述约束。	
A2	The most commonly used laser for generation femtosecond pulses employs titanium-doped sapphire (Ti:sapphire) crystal as its gain medium. It emits light with photon energy around 1.55 eV. Please find the expression of Gaussian pulse (Full-width at half maximum 35 fs in duration) out of such laser cavity. 最常用的产生飞秒脉冲的激光器采用掺钛蓝宝石（Ti:sapphire）晶体作为其增益介质。它发出光子能量约为 1.55 eV 的光。请求出该激光腔的高斯脉冲（半高宽为 35 fs）的表达式。	2 Points 2 分

A Gaussian pulse that is only dependent on time cannot propagate in real space. Consider a Gaussian pulse propagating in one dimensional *vacuum* along the z*-axis*.
仅依赖于时间的高斯脉冲无法在实空间中传播。考虑沿 z 轴在一维真空中传播的高斯脉冲。

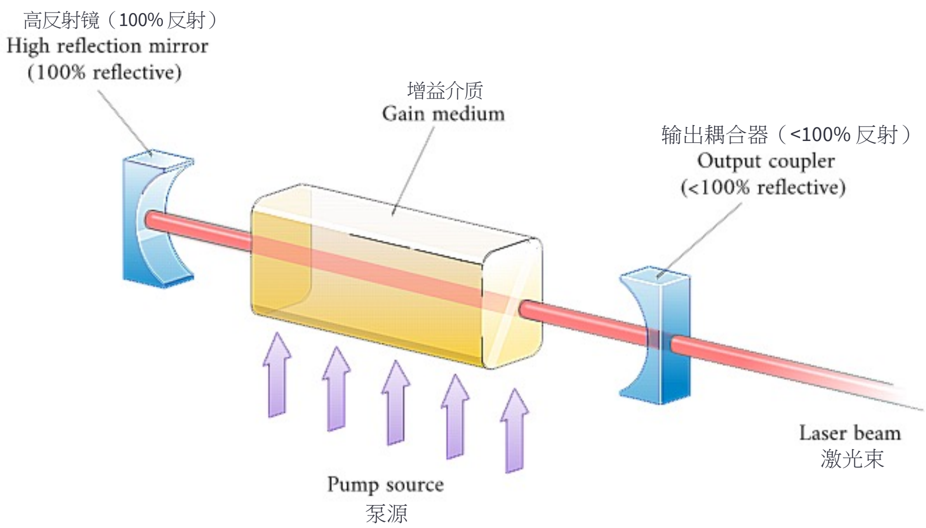
A3(1)	Please find the expression the electric field $E(z,t)$ of a Gaussian pulse and show that your answer can indeed propagate. *Hint: you may use the complex form to describe the propagating wave.* 请找出高斯脉冲电场的表达式 $E(z,t)$，并证明你的答案确实可以传播。 提示：您可以使用复数形式来描述传播波。	1 Points 1 分
A3(2)	Please find the expression of the electric field $E(t)$ if it is a Gaussian pulse after propagating at a distance L in medium of the frequency dependent refractive index $n(\omega)$. 如果是高斯脉冲，在有频率依赖的折射率 $n(\omega)$ 的介质中传播距离 L 后，求电场 $E(t)$ 的表达式。	1 Points 1 分

Part B: Dispersion 色散

Before entering the attosecond (10^{-18} s) regime, it historically took numerous effort of researchers to just generate femtosecond pulses (1 fs = 10^{-15}s), which now can be readily obtained from a standard Ti:sapphire laser and serve as the starting point to generate the even shorter attosecond pulse. One challenge at the time was to devise a laser cavity that can fight against the strong dispersion arise from traversing the Ti:sapphire crystal, an indispensable element in which amplification takes place. The dispersion here means the frequency dependent refractive index in the Ti:sapphire crystal, which has a strong absorption at 2.5 eV.

在实验技术能够进入阿秒尺度（10^{-18}s）之前，研究人员在上世纪付出了大量努力研究如何产生飞秒脉冲（1 fs = 10^{-15}s）。从而，使得我们现在可以较为容易地从钛蓝宝石激光器获得飞秒脉冲。而飞秒脉冲激光是产生更短的阿秒脉冲的起始光源。获得稳定飞秒脉冲输出需要克服的一大挑战是激光谐振腔内增益介质（钛蓝宝石晶体）带来的强色散效应。这里的色散是指 Ti:蓝宝石晶体中与频率相关的折射率，在 2.5 eV 对应的能量处有很强的吸收。

B1	Assuming the transition responsible for the absorption can be described with a classical model for an oscillating bound electron (mass m) oscillating at a characteristic frequency (Ω_0) about a nucleus. In the presence of an external AC E-field of amplitude E_0 oscillating at single frequency ω, please find the largest possible displacement of the electron. The damping force of the oscillator can be described by $f_d = -m\gamma v$ where v is the velocity of the oscillator and γ is a single parameter to describe the total effect from energy loss of all kinds. 假设电子对 2.5eV 的光吸收跃迁，可被以特征频率 Ω_0 围绕原子核振荡的束缚电子（质量 m）这一经典模型来描述。在外部交流电场（振幅为E_0，频率为ω）驱动的情况下，请找出电子的最大可能位移。电子受迫振动过程的阻尼力可以用$f_d = -m\gamma v$来描述，其中 v 是振子的速度，γ 是描述各种能量损失的总影响的单个参数。	4 Points 4 分
B2	Suppose the particle density of absorptive Ti^{3+} centers is N. Find the polarization density P and the corresponding dielectric function $\varepsilon(\omega)$, where $P = \varepsilon_0(\varepsilon_r - 1)E$. Make a drawing of the frequency dependent dielectric function $\varepsilon(\omega)$ and index of refraction $n(\omega)$ from zero frequency all the way through $2\Omega_0$. 假设吸收中心 Ti^{3+} 的粒子密度为N，求极化密度P及其对应的介电函数$\varepsilon(\omega)$，其中$P = \varepsilon_0\,(\varepsilon_r - 1)E$。请绘制频率依赖的介电函数 $\varepsilon(\omega)$ 以及折射率 $n(\omega)$，范围从零频率到至少 $2\Omega_0$。	4 Points 4 分



Part C: Pulse broadening effect 脉冲展宽效应

A Gaussian pulse, of which the transient frequency is constant in the time domain, is referred to as a Fourier-transform-limited pulse (τ_p) and has the shortest possible duration at a given bandwidth. After propagating through a dispersive medium of distance L, a transform-limited pulse will acquire a broadened pulse. It happens because the lasing frequency is close to the strong absorption line of the gain medium at 2.5 eV, and the high-order dispersion (frequency dependent part in refractive index $n(\omega)$) will start to take effect. (Hint: *The dissipative part during propagation can be ignored.*)

In this part, it is useful to define the propagating function

$$\beta(\omega) \equiv n(\omega)\frac{\omega}{c} \approx \beta_0 + \beta_1(\omega-\omega_0) + \beta_2(\omega-\omega_0)^2 + \cdots$$

which can be expanded around the center frequency ω_0 of the Gaussian pulse.

在时间域内载波具有恒定瞬时频率的高斯脉冲，通常被称为傅立叶变换极限脉冲，是在给定带宽下所能达到的最短的脉宽 τ_p 。变换极限脉冲在色散介质中传播距离 L 后，将获得展宽的脉冲。发生这种情况是因为激光频率接近 2.5 eV 增益介质的强吸收线，从而导致高阶色散效应（折射率 $n(\omega)$ 中与频率相关的部分）。(提示：传播过程中的耗散部分可以忽略。)

此处，如果定义一个随频率变化的传播函数并在中心频率ω_0附近做展开

$$\beta(\omega) \equiv n(\omega)\frac{\omega}{c} \approx \beta_0 + \beta_1(\omega-\omega_0) + \beta_2(\omega-\omega_0)^2 + \cdots$$

会对计算有所帮助。

C1	Show that the transform-limited pulse will broaden to a duration $\tau = \tau_p\sqrt{1+\frac{(\alpha\beta_2 L)^2}{\tau_p^4}}$, where α is a constant, β_2 is the second-order propagation factor from expansion about the center frequency ω_0. (*You may ignore effect from third-order and above*) 证明一个变换极限脉冲在传播距离L后，其脉冲宽度将变为 $\tau = \tau_p\sqrt{1+\frac{(\alpha\beta_2 L)^2}{\tau_p^4}}$，其中 α 是常数，β_2 是多项式展开的二阶传播因子。(请忽略三阶以上传播因子的贡献)	4 Points 5 分
C2	For an ultrafast femtosecond laser to oscillate in its stationary state, please come up with a design to compensate for such pulse broadening effect. 为了使超快飞秒激光器在形成稳态下振荡，请提出一种共振腔设计来补偿这种脉冲展宽效应。	2 Points 2 分

Part D: High-harmonic generation of attosecond pulse （高次谐波产生阿秒脉冲）

Take a short femtosecond pulse from Ti:sapphire laser (center frequency ω_0) and focus it into a gas medium. One could generate light at integer multiples of the driving frequency, often referred to as **high-harmonic generation (HHG).**

从钛宝石激光器（中心频率 ω_0 ）获取高能量飞秒脉冲并将其聚焦到气体介质中，可以在ω_0的整数倍频率上产生光辐

射，通常称为高次谐波产生（HHG）。

D1	One critical step for HHG is to drive the bound charge of molecules in the non-perturbative regime to initiate the *ionization dynamics*. If we take hydrogen molecules for HHG, please estimate the peak electric field strength required to initiate the process. HHG 的一个关键步骤是在非微扰状态下驱动分子的束缚电荷, 以驱动电离动力过程。如果我们采用氢分子作为 HHG，请估计有效驱动电离过程所需的峰值电场强度。	2 Points 2 分
D2	To allow radiation at new frequencies, please modify the oscillator model correspondingly and show your modification is viable. 为了产生新的辐射频率，请相应地修改前面的振子模型并证明您的修改是可行的。	4 Points 4 分
D3	Show how one could leverage the high-harmonic field to generate attosecond pulse. 请展示如何利用高次谐波场来生成相较于飞秒脉冲更短的阿秒脉冲。	2 Points 2 分

Solution:

A1.

$$a(t) = E_0 e^{-\frac{t^2}{\tau_p^2}} \cos(\omega_0 t + \varphi_0) \quad \text{(Real form)}$$

$$a(t) = E_0 e^{-\frac{t^2}{\tau_p^2}} e^{i(\omega_0 t + \varphi_0)} \quad \text{(Complex form)}$$

The average electric field

$$\langle a(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E_0 e^{-\frac{t^2}{\tau_p^2}} e^{i(\omega_0 t+\varphi_0)} dt = \lim_{T\to\infty} \frac{E_0}{2T} \int_{-T}^{T} dt \left(\int_{-\infty}^{\infty} \frac{2\sqrt{\pi}}{\tau_P} e^{-\frac{\tau_p^2 \omega^2}{4}} e^{-i\omega t} e^{i(\omega_0 t+\varphi_0)} d\omega \right)$$

$$= \lim_{T\to\infty} \frac{E_0}{2T} \int_{-\infty}^{\infty} d\omega \frac{2\sqrt{\pi}}{\tau_P} e^{-\frac{\tau_p^2 \omega^2}{4}} \left(\int_{-T}^{T} e^{-i\omega t} e^{i(\omega_0 t+\varphi_0)} dt \right) = 0$$

Here, we apply the identity,

$$\int_{-\infty}^{\infty} e^{-a\omega^2} e^{-i\omega t} d\omega = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{t^2}{4a}\right)$$

A2

$$\omega_0 = 2\pi f = 2\pi \frac{E_{ph}}{h} = 2\pi \times 375 \times 10^{12} \text{Hz}$$

FWHM is $35fs = 35 \times 10^{-15}$s,

$$\Rightarrow e^{-\frac{35fs^2}{\tau_p^2}} = \frac{1}{2} \Rightarrow \sqrt{\ln 2} = \frac{35fs}{\tau_p} \Rightarrow \tau_p = 4.2 \times 10^{-12}\text{s}$$

$$a(t) = E_0 e^{-\frac{t^2}{(4.2\times10^{-12}\text{s})^2}} \cos(2\pi \times 375 \times 10^{12}\text{Hz} \times t + \varphi_0)$$

$$\int_{-\infty}^{\infty} e^{-a\omega^2} e^{-i\omega t} d\omega = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{t^2}{4a}\right)$$

A3(1)

$$E(z,t) = E_0 \int_{-\infty}^{\infty} \exp\left(-\frac{\tau_p^2(\omega-\omega_0)^2}{4}\right) e^{i\left(\omega t - \frac{\omega}{c}z\right)}\, d\omega$$

where $\omega_0 = ck$

$$\frac{\partial^2 E}{\partial z^2} = E_0 \int_{-\infty}^{\infty} \left(-\frac{\omega^2}{c^2}\right) \exp\left(-\frac{\tau_p^2(\omega-\omega_0)^2}{4}\right) e^{i\left(\omega t - \frac{\omega}{c}z\right)}\, d\omega$$

$$\frac{\partial^2 E(z,t)}{\partial t^2} = -E_0 \int_{-\infty}^{\infty} \omega^2 \exp\left(-\frac{\tau_p^2(\omega-\omega_0)^2}{4}\right) e^{i(\omega t - kz)}\, d\omega$$

$$\Rightarrow \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = E_0 \int_{-\infty}^{\infty} \left(\frac{\omega^2}{c^2} - \frac{\omega^2}{c^2}\right) \exp\left(-\frac{\tau_p^2(\omega-\omega_0)^2}{4}\right) e^{i\left(\omega t - \frac{\omega}{c}z\right)}\, d\omega = 0$$

Which satisfies the wave equation.

A3(2)

$$E(t) = E_0 \int_{-\infty}^{\infty} \exp\left(-\frac{\tau_p^2(\omega-\omega_0)^2}{4}\right) e^{i\left(\omega t - n\frac{\omega}{c}L\right)}\, d\omega$$

B1: Equation of motion for harmonic oscillator

$$m\ddot{x} + m\gamma\dot{x} + m\Omega_0^2 x = -eE_0 e^{i\omega t}$$

Trail solution: $x(t) = X_0 e^{i\omega t}$

$$\Rightarrow (-m\omega^2 + i\omega m\gamma + m\Omega_0^2)X_0 = -eE_0$$

Therefore, the largest possible displacement of the electron is

$$\Rightarrow X_0 = \frac{1}{m(\omega^2 - \Omega_0^2) - i\omega m\gamma} eE_0$$

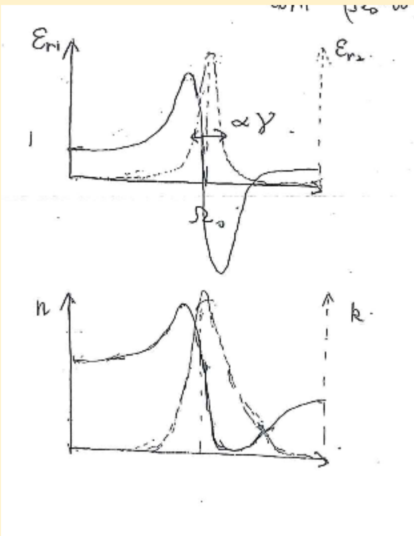
B2: Polarization is defined as

$$\vec{P} = -ex(t) = -\frac{1}{m(\omega^2 - \Omega_0^2) - i\omega m\gamma} Ne^2 E_0 e^{i\omega t} = \left(\frac{Ne^2}{m}\right)\frac{1}{(\Omega_0^2 - \omega^2) + i\gamma\omega}\vec{E}(t)$$

Dielectric function is given by

$$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E}$$

$$\Rightarrow \epsilon_r = 1 + \left(\frac{Ne^2}{\epsilon_0 m}\right)\frac{1}{(\Omega_0^2 - \omega^2) + i\gamma\omega}$$



C1: Given the propagation function

$$\beta(\omega) \equiv n(\omega)\frac{\omega}{c} \approx \beta_0 + \beta_1(\omega - \omega_0) + \beta_2(\omega - \omega_0)^2 + \cdots$$

$$a_{out}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a_{in}(\omega)e^{i\omega t}e^{-i\beta(\omega)L}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} a_{in}(\omega)e^{i(\omega_0 t-\beta_0 L)}e^{-i(\omega-\omega_0)t}e^{-iL(\beta_1(\omega-\omega_0)+\beta_2(\omega-\omega_0)^2)}\,d\omega$$

$$a_{out}(t) = e^{i(\omega_0 t-\beta_0 L)}\frac{E_0}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{\tau_p^2(\omega-\omega_0)^2}{4}\right)e^{-i(\omega-\omega_0)t}e^{-iL(\beta_1(\omega-\omega_0)+\beta_2(\omega-\omega_0)^2)}\,d\omega$$

Define $\omega' = \omega - \omega_0$,

$$a_{out}(t) = \frac{E_0}{2\pi}e^{i(\omega_0 t-\beta_0 L)}\int_{-\infty}^{\infty} \exp\left(-\frac{\tau_p^2\omega'^2}{4}\right)e^{-i\omega'}e^{-iL\left(\beta_1\omega'+\beta_2\omega'^2\right)}\,d\omega'$$

$$= \frac{E_0}{2\pi}e^{i(\omega_0 t-\beta_0 L)}\int_{-\infty}^{\infty} \exp\left(-\frac{\tau_p^2\omega'^2}{4}\right)e^{-i\omega' t}e^{-iL\left(\beta_1\omega'+\beta_2\omega'^2\right)}\,d\omega'$$

$$= \frac{E_0}{2\pi}e^{i(\omega_0 t-\beta_0 L)}\int_{-\infty}^{\infty} \exp\left(-\left(\frac{\tau_p^2}{4}+iL\beta_2\right)\omega'^2\right)e^{-i(t+L\beta_1)\omega'}\,d\omega'$$

$$= \frac{E_0}{2\pi}e^{i(\omega_0 t-\beta_0 L)}\int_{-\infty}^{\infty} \exp(-a\omega'^2)\,e^{-ib\omega'}d\omega'$$

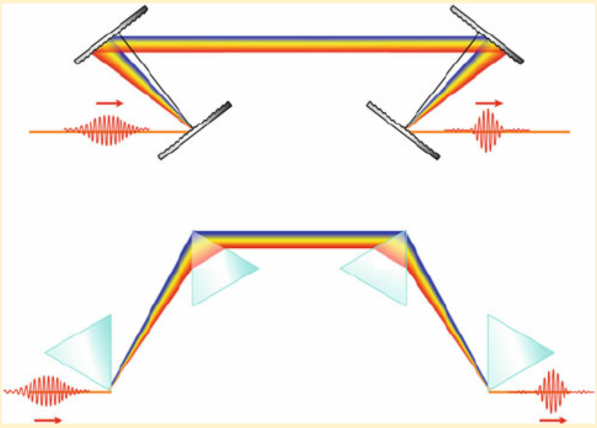
$$= \frac{E_0}{2\pi}e^{i(\omega_0 t-\beta_0 L)}\int_{-\infty}^{\infty} \exp\left(-\left(\sqrt{a}\omega'+\frac{ib}{2\sqrt{a}}\right)^2\right)e^{-\frac{b^2}{4a}}d\omega'$$

$$\propto e^{i(\omega_0 t-\beta_0 L)}\exp\left(-\frac{b^2}{4a}\right) \propto \exp(i\omega_0 t)\exp\left(-\frac{(t+L\beta_1)^2}{(\tau_p^2+4iL\beta_2)}\right)$$

$$\frac{(t+L\beta_1)^2}{(\tau_p^2+4iL\beta_2)} \sim \frac{t^2(\tau_p^2-4iL\beta_2)}{(\tau_p^4+(16L^2\beta_2^2))} \sim \frac{t^2}{\tau_p^2+\frac{16L^2\beta_2^2}{\tau_p^2}} + \text{oscillating terms}$$

$$\Rightarrow \tau_{out} \approx \tau_p\sqrt{1+\frac{16\beta_2^2L^2}{\tau_p^4}}$$

C2: Commonly used "insertion device" is prism compressor (lower panel). Grating compressor (upper panel), in principle, can work as well if the diffraction grating can be optimized to near-unity. Full score of 2 points gives to any one of the two scheme or their equivalence.



D1: For ionization to occur, momentum acquired by particle in one cycle of field should exceed that converted from work function of bound charge (-13.6eV),

$$\frac{eE}{\omega} \sim \text{momentum acquired by particle in one cycle of electric field} > \sqrt{2mW}$$

$$\Rightarrow E > \omega\frac{\sqrt{2mW}}{e} \approx 3\times 10^{10}\,\frac{V}{m} = 30\,\frac{GV}{m}$$

Remark: We can define the Keldysh parameter where $\gamma \equiv \omega\frac{\sqrt{2mW}}{eE} < 1$

D2: In the strong field limit, the oscillation becomes anharmonic as the potential energy of bound charge follows the model of anharmonic oscillator,

$$u(x) = \alpha_1 x^2 + \alpha_2 x^3 + \cdots$$
$$m\ddot{x} = -2\alpha_1 x - 3\alpha_2 x^2 + eE_0 e^{i\omega_0 t} + O(x^4)$$
$$m\ddot{x} + \alpha_1' x = \alpha_2' x^2 + eE_0 e^{i\omega_0 t}$$

First order:

$$m\ddot{x} + \alpha_1' x = eE_0 e^{i\omega_0 t} \Rightarrow x_1 \propto E_0 e^{i\omega_0 t}$$

2nd order:

$$x = x_1 + x_2\ (x_2 \ll x_1)$$
$$m(\ddot{x_1} + \ddot{x_2}) + \alpha_1'(x_1 + x_2) = \alpha_2'(x_1 + x_2)^2 + \cdots = \alpha_2'(x_1^2 + 2x_1x_2 + x_2^2) + \cdots$$
$$\Rightarrow m\ddot{x_2} + \alpha_1' x_2 \approx \alpha_2' x_1^2$$
$$\Rightarrow x_2 \propto x_1^2 \propto E_0^2 e^{i2\omega_0 t}$$

The stronger the field is, the high order terms gets more pronounced. In general, we can show that

$$x_n \propto E_0^n e^{in\omega_0 t}$$

D3: If oscillator is anharmonic, the n-th order dipole is $\propto E_0^n e^{in\omega_0 t}$,

Suppose $E_0 \propto e^{-\frac{t^2}{\tau_p^2}}$,

$$\Rightarrow P^{(n)} \propto e^{-\frac{nt^2}{\tau_p^2}} e^{in\omega_0 t} = e^{-\frac{t^2}{\left(\frac{\tau_p}{\sqrt{n}}\right)^2}} e^{in\omega_0 t}$$

~ ENF OF Paper 2 卷二完 ~